FUNDAMENTAL CONCEPTS IN FINANCIAL MANAGEMENT

PART 2

CHAPTER 2

TIME VALUE OF MONEY

Will You Be Able to Retire?

Your reaction to this question is probably, "First things first! I'm worried about getting a job, not about retiring!" However, understanding the retirement situation could help you land a job, because (1) this is an important issue today, (2) employers like to hire people who know what's happening in the real world, and (3) professors often test on the time value of money with problems related to saving for future purposes, including retirement.

A recent *Fortune* article began with some interesting facts: (1) The U.S. savings rate is the lowest of any industrial nation. (2) The ratio of U.S. workers to retirees, which was 17 to 1 in 1950, is now down to 3 to 1, and it will decline to less than 2 to 1 after 2020. (3) With so few people paying into the Social Security system and so many drawing funds out, Social Security is going to be in serious trouble. The article concluded that even people making $85,000 per year will have trouble maintaining a reasonable standard of living after they retire, and many of today's college students will have to support their parents.

This is an important issue for millions of Americans, but many don't know how to deal with it. When *Fortune* studied the retirement issue, using the tools and techniques described in this chapter, they concluded that most Americans have been putting their heads in the sand, ignoring what is almost certainly going to be a huge personal and social problem. However, if you study this chapter carefully, you can avoid the trap that is likely to catch so many people.

Excellent retirement calculators are available at **http://www.ssa.gov** *and* **http://www.choosetosave.org/calculators**. *These calculators allow you to input hypothetical retirement savings information, and the program shows if current retirement savings will be sufficient to meet retirement needs.*

Putting Things In Perspective

Time value analysis has many applications, including planning for retirement, valuing stocks and bonds, setting up loan payment schedules, and making corporate decisions regarding investing in new plant and equipment. *In fact, of all financial concepts, time value of money is the single most important.*

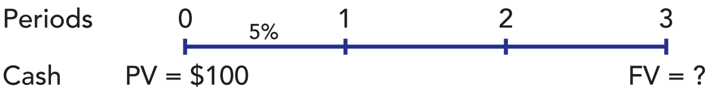
Indeed, time value analysis is used throughout the book, so it is vital that you understand this chapter before continuing.

You need to understand basic time value concepts, but conceptual knowledge will do you little good if you can't do the required calculations. Therefore, this chapter is heavy on calculations. Also, most students studying finance have a financial or scientific calculator, and some also own or have access to a computer. Moreover, one of these tools is necessary to work many finance problems in a "reasonable" length of time. However, when they start on this chapter, many students don't know how to use the time value functions in their calculator or computer. If you are in that situation, you will find yourself simultaneously studying concepts and trying to learn to use your calculator, and you will need more time to cover this chapter than you might expect.[1]

2.1 TIME LINES

The first step in time value analysis is to set up a **time line,** which will help you visualize what's happening in a particular problem. To illustrate, consider the following diagram, where PV represents $100 that is on hand today and FV is the value that will be in the account on a future date:

Time Line
An important tool used in time value analysis; it is a graphical representation used to show the timing of cash flows.



The intervals from 0 to 1, 1 to 2, and 2 to 3 are time periods such as years or months. Time 0 is today, and it is the beginning of Period 1; Time 1 is one period from today, and it is both the end of Period 1 and the beginning of Period 2; and so on. Although the periods are often years, periods can also be quarters or months or even days. Note that each tick mark corresponds to both the *end* of one period and the *beginning* of the next one. Thus, if the periods are years, the tick mark at Time 2 represents both the *end* of Year 2 and the *beginning* of Year 3.

Cash flows are shown directly below the tick marks, and the relevant interest rate is shown just above the time line. Unknown cash flows, which you are trying to find, are indicated by question marks. Here the interest rate is 5 percent; a single cash outflow, $100, is invested at Time 0; and the Time 3 value is an unknown inflow. In this example, cash flows occur only at Times 0 and 3, with no flows at Times 1 or 2. Note that in our example the interest rate is constant for all three years. That condition is generally true, but if it were not then we would show different interest rates for the different periods.

Time lines are essential when you are first learning time value concepts, but even experts use them to analyze complex finance problems, and we use them throughout the book. We begin each problem by setting up a time line to show what's happening, after which we provide an equation that must be solved to find the answer, and then we explain how to use a regular calculator, a financial calculator, and a spreadsheet to find the answer.

[1] Calculator manuals tend to be long and complicated, partly because they cover a number of topics that aren't required in the basic finance course. Therefore, we provide, on the ThomsonNOW Web site, tutorials for the most commonly used calculators. The tutorials are keyed to this chapter, and they show exactly how to do the required calculations. If you don't know how to use your calculator, go to the ThomsonNOW Web site, get the relevant tutorial, and go through it as you study the chapter.

Do time lines deal only with years or could other periods be used?

Set up a time line to illustrate the following situation: You currently have \$2,000 in a three-year certificate of deposit (CD) that pays a guaranteed 4 percent annually.

2.2 FUTURE VALUES

Future Value (FV)
The amount to which a cash flow or series of cash flows will grow over a given period of time when compounded at a given interest rate.

Present Value (PV)
The value today of a future cash flow or series of cash flows.

Compounding
The arithmetic process of determining the final value of a cash flow or series of cash flows when compound interest is applied.

A dollar in hand today is worth more than a dollar to be received in the future because, if you had it now, you could invest it, earn interest, and end up with more than a dollar in the future. The process of going to **future values (FVs)** from **present values (PVs)** is called **compounding.** To illustrate, refer back to our three-year time line and assume that you plan to deposit \$100 in a bank that pays a guaranteed 5 percent interest each year. How much would you have at the end of Year 3? We first define some terms, after which we set up a time line and show how the future value is calculated.

PV = Present value, or beginning amount. In our example, PV = \$100.

FV_N = Future value, or ending amount, of your account after N periods. Whereas PV is the value now, or the *present value*, FV_N is the value N periods into the *future*, after the interest earned has been added to the account.

CF_t = Cash flow. Cash flows can be positive or negative. The cash flow for a particular period is often given a subscript, CF_t, where t is the period. Thus, CF_0 = PV = the cash flow at Time 0, whereas CF_3 would be the cash flow at the end of Period 3.

I = Interest rate earned per year. Sometimes a lowercase "i" is used. Interest earned is based on the balance at the beginning of each year, and we assume that it is paid at the end of the year. Here I = 5%, or, expressed as a decimal, 0.05. Throughout this chapter, we designate the interest rate as I because that symbol (or I/YR, for interest rate per year) is used on most financial calculators. Note, though, that in later chapters we use the symbol "r" to denote rates because r (for rate of return) is used more often in the finance literature. Note too that in this chapter we generally assume that interest payments are guaranteed by the U.S. government, hence they are certain. In later chapters we will consider risky investments, where the interest rate actually earned might differ from its expected level.

INT = Dollars of interest earned during the year = Beginning amount × I. In our example, INT = \$100(0.05) = \$5.

N = Number of periods involved in the analysis. In our example N = 3. Sometimes the number of periods is designated with a lowercase "n," so both N and n indicate number of periods.

We can use four different procedures to solve time value problems.[2] These methods are described in the following sections.

[2] A fifth procedure, using tables that show "interest factors," was used before financial calculators and computers became available. Now, though, calculators and spreadsheets such as *Excel* are programmed to calculate the specific factor needed for a given problem and then to use it to find the FV. This is much more efficient than using the tables. Moreover, calculators and spreadsheets can handle fractional periods and fractional interest rates, like the FV of \$100 after 3.75 years when the interest rate is 5.375 percent, whereas tables provide numbers only for specific periods and rates. For these reasons, tables are not used in business today; hence we do not discuss them in the text.

Simple versus Compound Interest

As noted in the text, when interest is earned on the interest earned in prior periods, as was true in our example and is always true when we apply Equation 2-1, this is called **compound interest.** If interest is not earned on interest, then we have **simple interest.** The formula for FV with simple interest is FV = PV + PV(I)(N), so in our example FV would have been $100 + $100(0.05)(3) = $100 + $15 = $115 based on simple interest. Most financial contracts are based on compound interest, but in legal proceedings the law often specifies that simple interest must be used. For example, Maris Distributing, a company founded by home run king Roger Maris, won a lawsuit against Anheuser-Busch (A-B) because A-B had breached a contract and taken away Maris's franchise to sell Budweiser beer. The judge awarded Maris $50 million plus interest at 10 percent from 1997 (when A-B breached the contract) until the payment is actually made. The interest award was based on simple interest, which as of 2004 had raised the total from $50 million to $50 million + 0.10($50 million)(7 years) = $85 million. If the law had allowed compound interest, the award would have totaled ($50 million)$(1.10)^7$ = $97.44 million, or $12.44 million more. This legal procedure dates back to the days before we had calculators and computers. The law moves slowly!

Compound Interest
Occurs when interest is earned on prior periods' interest.

Simple Interest
Occurs when interest is not earned on interest.

1. Step-by-Step Approach

The time line used to find the FV of $100 compounded for three years at 5 percent, along with some calculations, is shown below:

Multiply the initial amount, and each succeeding amount, by (1 + I) = (1.05):

Time	0	1	2	3
	5%			
Amount at beginning of period	$100.00 ----►	$105.00 ----►	$110.25 ----►	$115.76

You start with $100 in the account—this is shown at t = 0:

- You earn $100(0.05) = $5 of interest during the first year, so the amount at the end of Year 1 (or t = 1) is $100 + $5 = $105.
- You begin the second year with $105, earn 0.05($105) = $5.25 on the now larger beginning-of-period amount, and end the year with $110.25. Interest during Year 2 is $5.25, and it is higher than the first year's interest, $5, because you earned $5(0.05) = $0.25 interest on the first year's interest. This is called "compounding," and when interest is earned on interest, this is called "compound interest."
- This process continues, and because the beginning balance is higher in each successive year, the interest earned each year increases.
- The total interest earned, $15.76, is reflected in the final balance, $115.76.

The step-by-step approach is useful because it shows exactly what is happening. However, this approach is time-consuming, especially if a number of years are involved, so streamlined procedures have been developed.

2. Formula Approach

In the step-by-step approach, we multiply the amount at the beginning of each period by (1 + I) = (1.05). If N = 3, then we multiply by (1 + I) three different times, which is the same as multiplying the beginning amount by $(1 + I)^3$. This concept can be extended, and the result is this key equation:

$$FV_N = PV(1 + I)^N \tag{2-1}$$

We can apply Equation 2-1 to find the FV in our example:

$$FV_3 = \$100(1.05)^3 = \$115.76$$

Equation 2-1 can be used with any calculator that has an exponential function, making it easy to find FVs, no matter how many years are involved.

3. Financial Calculators

Financial calculators are extremely helpful when working time value problems. Their manuals explain calculators in detail, and we provide summaries of the features needed to work the problems in this book for several popular calculators on the ThomsonNOW Web site. Also, see the box entitled "Hints on Using Financial Calculators" for suggestions that will help you avoid some common mistakes. If you are not yet familiar with your calculator, we recommend that you go through our tutorial as you study this chapter.

First, note that financial calculators have five keys that correspond to the five variables in the basic time value equations. We show the inputs for our example above the keys and the output, the FV, below its key. Because there are no periodic payments, we enter 0 for PMT. We describe the keys in more detail below the diagram.

3	5	–100	0	
N	I/YR	PV	PMT	FV
				115.76

N = Number of periods. Some calculators use n rather than N.

I/YR = Interest rate per period. Some calculators use i or I rather than I/YR.

PV = Present value. In our example we begin by making a deposit, which is an outflow, so the PV should be entered with a negative sign. On most calculators you must enter the 100, then press the +/− key to switch from +100 to −100. If you enter −100 directly, this will subtract 100 from the last number in the calculator and give you an incorrect answer.

PMT = Payment. This key is used if we have a series of equal, or constant, payments. Because there are no such payments in our illustrative problem, we enter PMT = 0. We will use the PMT key when we discuss annuities later in this chapter.

FV = Future value. In this example, the FV is positive because we entered the PV as a negative number. If we had entered the 100 as a positive number, then the FV would have been negative.

As noted in our example, you first enter the known values (N, I/YR, PMT, and PV) and then press the FV key to get the answer, 115.76. Again, note that if you entered the PV as 100 without a minus sign, the FV would be given as a negative. The calculator *assumes* that either the PV or the FV is negative. This should not be confusing if you think about what you are doing.

4. Spreadsheets[3]

Students generally use calculators for homework and exam problems, but in business people generally use spreadsheets for problems that involve the time

[3] If you have never worked with spreadsheets, you might want to skip this section. However, you might want to go through it and refer to this chapter's *Excel* model to get an idea of how spreadsheets work.

Hints on Using Financial Calculators

When using a financial calculator, make sure your machine is set up as indicated below. Refer to your calculator manual or to our calculator tutorial on the ThomsonNOW Web site for information on setting up your calculator.

- **One payment per period.** Many calculators "come out of the box" assuming that 12 payments are made per year; that is, they assume monthly payments. However, in this book we generally deal with problems where only one payment is made each year. *Therefore, you should set your calculator at one payment per year and leave it there. See our tutorial or your calculator manual if you need assistance.*
- **End mode.** With most contracts, payments are made at the end of each period. However, some contracts call for payments at the beginning of each period. You can switch between "End Mode" and "Begin Mode," depending on the problem you are solving. *Because most of the problems in this book call for end-of-period payments, you should return your calculator to End Mode after you work a problem where payments are made at the beginning of periods.*
- **Negative sign for outflows.** *Outflows must be entered as negative numbers. This generally means typing the outflow as a positive number and then pressing the +/– key to convert from + to – before hitting the enter key.*
- **Decimal places.** With most calculators, you can specify from 0 to 11 decimal places. When working with dollars, we generally specify two decimal places. When dealing with interest rates, we generally specify two places if the rate is expressed as a percentage, like 5.25 percent, but we specify four places if the rate is expressed as a decimal, like 0.0525.
- **Interest rates.** *For arithmetic operations with a nonfinancial calculator, the 0.0525 must be used, but with a financial calculator you must enter 5.25, not .0525, because financial calculators assume that rates are stated as percentages.*

value of money (TVM). Spreadsheets show in detail what is happening, and they help us reduce both conceptual and data-entry errors. The spreadsheet discussion can be skipped without loss of continuity, but if you understand the basics of *Excel* and have access to a computer, we recommend that you go through this section. Even if you aren't familiar with spreadsheets, our discussion will still give you an idea of how they operate.

We used *Excel* to create Table 2-1, which summarizes the four methods for finding the FV and shows the spreadsheet formulas toward the bottom. Note that spreadsheets can be used to do calculations, but they can also be used like a word processor to create exhibits like Table 2-1, which includes text, drawings, and calculations. The letters across the top designate columns, the numbers to the left designate rows, and the rows and columns jointly designate cells. Thus, C14 is the cell where we specify the −\$100 investment, C15 shows the interest rate, and C16 shows the number of periods. We then created a time line on Rows 17 to 19, and on Row 21 we have *Excel* go through the step-by-step calculations, multiplying the beginning-of-year values by $(1 + I)$ to find the compounded value at the end of each period. Cell G21 shows the final result. Then, on Row 23, we illustrate the formula approach, using *Excel* to solve Equation 2-1 and find the FV, \$115.76. Next, on Rows 25 to 27, we show a picture of the calculator solution. Finally, on Rows 29 and 30 we use *Excel*'s built-in FV function to find the answers given in Cells G29 and G30. The G29 answer is based on fixed inputs while the G30 answer is based on cell references, which makes it easy to change inputs and see the effects on the output.

Table 2-1 demonstrates that all four methods get the same result, but they use different calculating procedures. It also shows that with *Excel* all the inputs

TABLE 2-1 *Summary of Future Value Calculations*

	A	B	C	D	E	F	G
14	Investment	$= CF_0 = PV =$	−$100.00				
15	Interest rate	= I =	5.00%				
16	No. of periods	= N =	3				
17			Periods:	0	1	2	3
18							
19			Cash Flow Time Line:	−$100 →	→	→	FV = ?
20							
21	Step-by-Step Approach:			$100 →	$105.00 →	$110.25 →	$115.76
22							
23	Formula Approach: $FV_N = PV(1+I)^N$			$FV_N =$	$\$100(1.05)^3$	=	$115.76
24							
25			3	5	−$100.00	$0	
26	Calculator Approach:		N	I/YR	PV	PMT	FV
27							$115.76
28							
29	Excel Approach:		Fixed inputs:	$FV_N =$	=FV(0.05,3,0,−100) =		$115.76
30			Cell references:	$FV_N =$	=FV(C15,C16,0,C14) =		$115.76
31							
32	In the Excel formula, the terms are entered in this sequence: interest, periods, 0 to indicate no intermediate cash flows, and then the PV. The data can be entered as fixed numbers or as cell references.						

are shown in one place, which makes checking data entries relatively easy. Finally, it shows that *Excel* can be used to create exhibits, which are quite important in the real world. In business, it's often as important to explain what you are doing as it is to "get the right answer," because if decision makers don't understand your analysis, they may well reject your recommendations.

Graphic View of the Compounding Process

Figure 2-1 shows how a $1 investment grows over time at different interest rates. We made the curves by solving Equation 2-1 with different values for N and I. The interest rate is a growth rate: If a sum is deposited and earns 5 percent interest per year, then the funds on deposit will grow by 5 percent per year. Note also that time value concepts can be applied to anything that grows—sales, population, earnings per share, or your future salary.

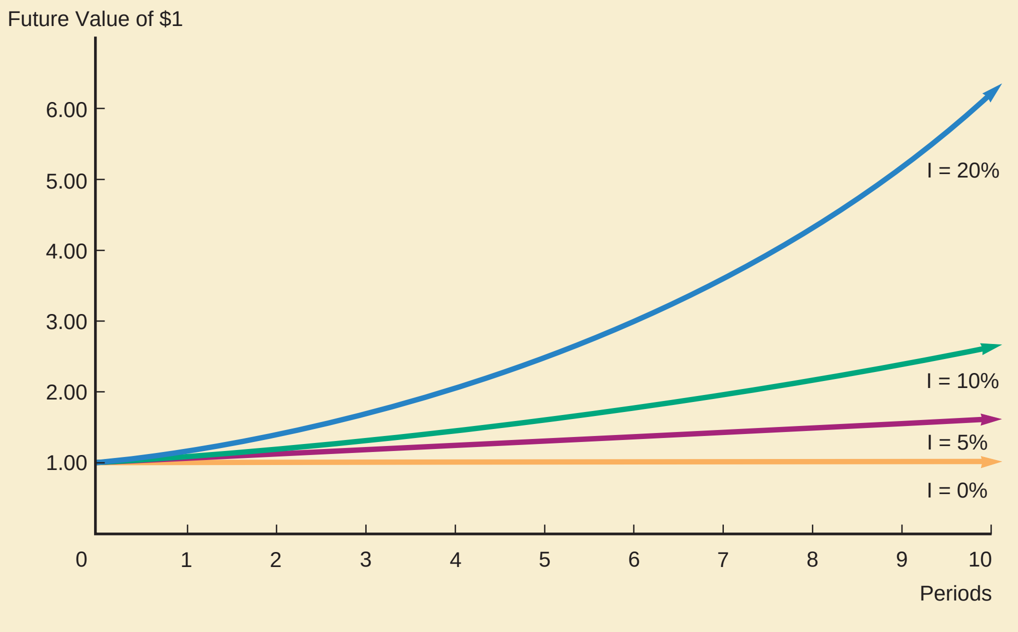
SELF TEST

Explain why this statement is true: "A dollar in hand today is worth more than a dollar to be received next year."

What is compounding? What's the difference between simple interest and compound interest? What would the future value of $100 be after five years at 10 percent *compound* interest? At 10 percent *simple* interest? ($161.05; $150.00)

Suppose you currently have $2,000 and plan to purchase a three-year certificate of deposit (CD) that pays 4 percent interest compounded annually. How much will you have when the CD matures?

FIGURE 2-1 *Growth of $1 at Various Interest Rates and Time Periods*



How would your answer change if the interest rate were 5 percent, or 6 percent, or 20 percent? ($2,249.73; $2,315.25; $2,382.03; $3,456.00) Hint: With a calculator, enter N = 3, I/YR = 4, PV = −2000, and PMT = 0, then press FV to get 2,249.73. Then, enter I/YR = 5 to override the 4 percent and press FV again to get the second answer. In general, you can change one input at a time to see how the output changes.

A company's sales in 2005 were $100 million. If sales grow at 8 percent, what will they be 10 years later, in 2015? ($215.89 million)

How much would $1, growing at 5 percent per year, be worth after 100 years? What would FV be if the growth rate were 10 percent? ($131.50; $13,780.61)

2.3 PRESENT VALUES

Finding a present value is the reverse of finding a future value. Indeed, we simply solve Equation 2-1, the formula for the future value, for the PV to produce the basic present value equation, 2-2:

$$\text{Future value} = FV_N = PV(1 + I)^N \tag{2-1}$$

$$\text{Present value} = PV = \frac{FV_N}{(1 + I)^N} \tag{2-2}$$

We illustrate PVs with the following example. A broker offers to sell you a Treasury bond that three years from now will pay $115.76. Banks are currently

Opportunity Cost
The rate of return you could earn on an alternative investment of similar risk.

offering a guaranteed 5 percent interest on three-year certificates of deposit (CDs), and if you don't buy the bond you will buy a CD. The 5 percent rate paid on the CDs is defined as your **opportunity cost,** or the rate of return you could earn on an alternative investment of similar risk. Given these conditions, what's the most you should pay for the bond? We answer this question using the four methods we discussed in the last section—step-by-step, formula, calculator, and spreadsheet. Table 2-2 summarizes our results.

First, recall from the future value example in the last section that if you invested $100 at 5 percent it would grow to $115.76 in three years. You would also have $115.76 after three years if you bought the T-bond. Therefore, the most you should pay for the bond is $100—this is its "fair price." If you could buy the bond for *less than* $100, you should buy it rather than invest in the CD. Conversely, if its price were *more than* $100, you should buy the CD. If the bond's price were exactly $100, you should be indifferent between the T-bond and the CD.

Discounting
The process of finding the present value of a cash flow or a series of cash flows; discounting is the reverse of compounding.

The $100 is defined as the present value, or PV, of $115.76 due in three years when the appropriate interest rate is 5 percent. In general, *the present value of a cash flow due N years in the future is the amount which, if it were on hand today, would grow to equal the given future amount*. Because $100 would grow to $115.76 in three years at a 5 percent interest rate, $100 is the present value of $115.76 due in three years at a 5 percent rate. Finding present values is called **discounting,** and as noted above, it is the reverse of compounding—if you know the PV, you can compound to find the FV, while if you know the FV, you can discount to find the PV.

The top section of Table 2-2 calculates the PV using the step-by-step approach. When we found the future value in the previous section, we worked from left to right, multiplying the initial amount and each subsequent amount

TABLE 2-2 *Summary of Present Value Calculations*

	A	B	C	D	E	F	G
64	Future payment = CF_N = FV =		$115.76				
65	Interest rate =	I =	5.00%				
66	No. of periods =	N =	3				
67			Periods:	0	1	2	3
68							
69			Cash Flow Time Line:	PV = ? ←	←	←	$115.76
70							
71	Step-by-Step Approach:			$100.00 ←	$105.00 ←	$110.25 ←	$115.76
72							
73	Formula Approach: $PV = FV_N / (1 + I)^N$			$PV=\$115.76/(1.05)^3$		=	$100.00
74			3	5		$0	$115.76
75	Calculator Approach:		N	I/YR	PV	PMT	FV
76					−$100.00		
77							
78	Excel Approach:		Fixed inputs:	PV =	=PV(0.05,3,0,115.76)	=	−$100.00
79			Cell references:	PV =	=PV(C65,C66,0,C64)	=	−$100.00
80							
81							
82	In the Excel formula, 0 indicates that there are no intermediate cash flows.						

by (1 + I). To find present values, we work backward, or from right to left, dividing the future value and each subsequent amount by (1 + I). This procedure shows exactly what's happening, and that can be quite useful when you are working complex problems. However, it's inefficient, especially if you are dealing with a number of years.

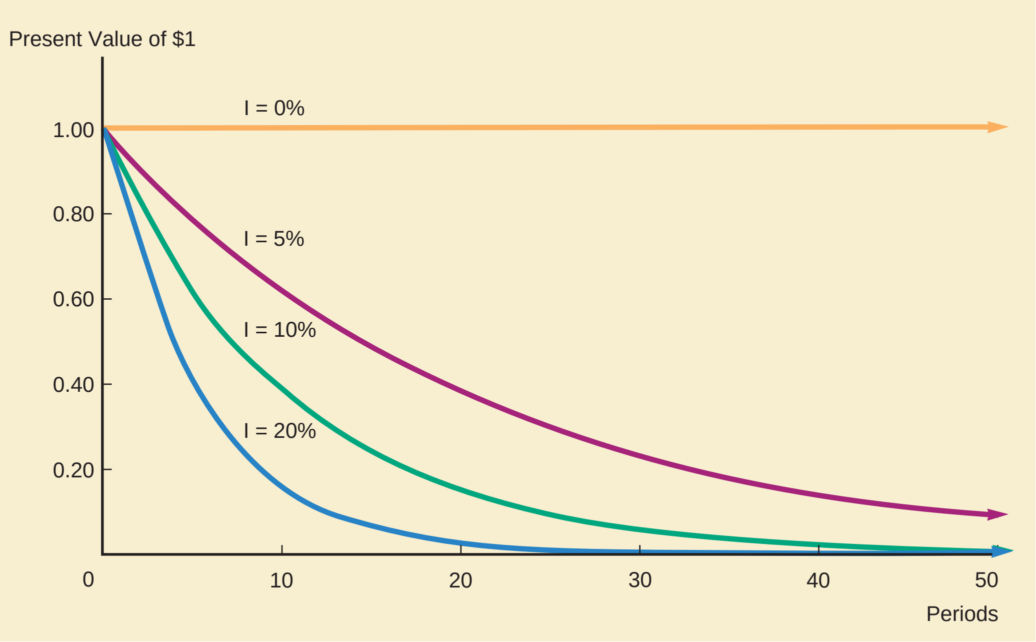
With the formula approach we use Equation 2-2, simply dividing the future value by $(1 + I)^N$. This is more efficient than the step-by-step approach, and it gives the same result. Equation 2-2 is built into financial calculators, and as shown in Table 2-2, we can find the PV by entering values for N, I/YR, PMT, and FV, and then pressing the PV key. Finally, spreadsheets have a function that's essentially the same as the calculator, which also solves Equation 2-2.

The fundamental goal of financial management is to maximize the firm's value, and the value of a business (or any asset, including stocks and bonds) is the *present value* of its expected future cash flows. Because present value lies at the heart of the valuation process, we will have much more to say about it in the remainder of this chapter and throughout the book.

Graphic View of the Discounting Process

Figure 2-2 shows that the present value of a sum to be received in the future decreases and approaches zero as the payment date is extended further and further into the future and also that the present value falls faster the higher the interest rate. At relatively high rates, funds due in the future are worth very little today, and even at relatively low rates present values of sums due in the very distant future are quite small. For example, at a 20 percent discount rate, $1 million due in 100 years would be worth only $0.0121 today. This is because $0.0121 would grow to $1 million in 100 years when compounded at 20 percent.

FIGURE 2-2 *Present Value of $1 at Various Interest Rates and Time Periods*



SELF TEST

What is "discounting," and how is it related to compounding? How is the future value equation (2-1) related to the present value equation (2-2)?

How does the present value of a future payment change as the time to receipt is lengthened? As the interest rate increases?

Suppose a U.S. government bond promises to pay $2,249.73 three years from now. If the going interest rate on three-year government bonds is 4 percent, how much is the bond worth today? How would your answer change if the bond matured in five rather than three years? What if the interest rate on the five-year bond were 6 percent rather than 4 percent? ($2,000; $1,849.11; $1,681.13)

How much would $1,000,000 due in 100 years be worth today if the discount rate were 5 percent? If the discount rate were 20 percent? ($7,604.49; $0.0121)

2.4 FINDING THE INTEREST RATE, I

Thus far we have used Equations 2-1 and 2-2 to find future and present values. Those equations have four variables, and if we know three of them, we can solve for the fourth. Thus, if we know PV, I, and N, then we can solve 2-1 for FV, while if we know FV, I, and N we can solve 2-2 to find PV. That's what we did in the preceding two sections.

Now suppose we know PV, FV, and N, and we want to find I. For example, suppose we know that a given bond has a cost of $100 and that it will return $150 after 10 years. Thus, we know PV, FV, and N, and we want to find the rate of return we will earn if we buy the bond. Here's the situation:

$$FV = PV(1 + I)^N$$
$$\$150 = \$100(1 + I)^{10}$$
$$\$150/\$100 = (1 + I)^{10}$$
$$1.5 = (1 + I)^{10}$$

Unfortunately, we can't factor I out to produce as simple a formula as we could for FV and PV—we can solve for I, but it requires a bit more algebra.[4] However, financial calculators and spreadsheets can find interest rates almost instantly. Here's the calculator setup:

10		−100	0	150
N	I/YR	PV	PMT	FV
	4.14			

Enter N = 10, PV = −100, PMT = 0 because there are no payments until the security matures, and FV = 150. Then, when you press the I/YR key, the calculator gives the answer, 4.14 percent. You would get this same answer with a spreadsheet.

[4] Raise the left side of the equation, the 1.5, to the power 1/N = 1/10 = 0.1, getting 1.0414. That number is 1 plus the interest rate, so the interest rate is 0.0414 = 4.14%.

The U.S. Treasury offers to sell you a bond for \$585.43. No payments will be made until the bond matures 10 years from now, at which time it will be redeemed for \$1,000. What interest rate would you earn if you bought this bond for \$585.43? What rate would you earn if you could buy the bond for \$550? For \$600? (5.5%; 6.16%; 5.24%)

Microsoft earned \$0.12 per share in 1994. Ten years later, in 2004, it earned \$1.04. What was the growth rate in Microsoft's earnings per share (EPS) over the 10-year period? If EPS in 2004 had been \$0.65 rather than \$1.04, what would the growth rate have been? (24.1%; 18.41%)

2.5 FINDING THE NUMBER OF YEARS, N

We sometimes need to know how long it will take to accumulate a given sum of money, given our beginning funds and the rate we will earn on those funds. For example, suppose we believe that we could retire comfortably if we had \$1 million, and we want to find how long it will take us to have \$1 million, assuming we now have \$500,000 invested at 4.5 percent. We cannot use a simple formula—the situation is like that with interest rates. We could set up a formula that uses logarithms, but calculators and spreadsheets can find N very quickly. Here's the calculator setup:

Enter I/YR = 4.5, PV = −500000, PMT = 0, and FV = 1000000. Then, when we press the N key, we get the answer, 15.7473 years. If you plug N = 15.7473 into the FV formula, you can prove that this is indeed the correct number of years:

$$FV = PV(1 + I)^N = \$500,000(1.045)^{15.7473} = \$1,000,000$$

We would also get N = 15.7473 with a spreadsheet.

How long would it take \$1,000 to double if it were invested in a bank that pays 6 percent per year? How long would it take if the rate were 10 percent? (11.9 years; 7.27 years)

Microsoft's 2004 earnings per share were \$1.04, and its growth rate during the prior 10 years was 24.1 percent per year. If that growth rate were maintained, how long would it take for Microsoft's EPS to double? (3.21 years)

2.6 ANNUITIES

Thus far we have dealt with single payments, or "lump sums." However, many assets provide a series of cash inflows over time, and many obligations like auto, student, and mortgage loans require a series of payments. If the payments are equal and are made at fixed intervals, then the series is an **annuity.** For example,

Annuity
A series of equal payments at fixed intervals for a specified number of periods.

Ordinary (Deferred) Annuity
An annuity whose payments occur at the end of each period.

Annuity Due
An annuity whose payments occur at the beginning *of each period.*

$100 paid at the end of each of the next three years is a three-year annuity. If the payments occur at the *end* of each year, then we have an **ordinary** (or **deferred**) **annuity.** If the payments are made at the *beginning* of each year, then we have an **annuity due.** Ordinary annuities are more common in finance, so when we use the term "annuity" in this book, assume that the payments occur at the ends of the periods unless otherwise noted.

Here are the time lines for a $100, three-year, 5 percent, ordinary annuity and for the same annuity on an annuity due basis. With the annuity due, each payment is shifted back to the left by one year. A $100 deposit will be made each year, so we show the payments with minus signs:

Ordinary Annuity:

Periods	0	1	2	3
	5%			
Payments		−$100	−$100	−$100

Annuity Due:

Periods	0	1	2	3
	5%			
Payments	−$100	−$100	−$100	

As we demonstrate in the following sections, we can find an annuity's future and present values, the interest rate built into annuity contracts, and how long it takes to reach a financial goal using an annuity. Keep in mind that annuities must have *constant payments* and a *fixed number of periods*. If these conditions don't hold, then we don't have an annuity.

SELF TEST

What's the difference between an ordinary annuity and an annuity due?

Why should you rather receive an annuity due for $10,000 per year for 10 years than an otherwise similar ordinary annuity?

2.7 FUTURE VALUE OF AN ORDINARY ANNUITY

The future value of an annuity can be found using the step-by-step approach or with a formula, a financial calculator, or a spreadsheet. To illustrate, consider the ordinary annuity diagrammed earlier, where you deposit $100 at the end of each year for three years and earn 5 percent per year. How much will you have at the end of the third year? The answer, $315.25, is defined as the future value of the annuity, $\mathbf{FVA_N}$, and it is shown in Table 2-3.

FVA_N
The future value of an annuity over N periods.

As shown in the step-by-step section of the table, we compound each payment out to Time 3, then sum those compounded values to find the annuity's FV, FVA_3 = $315.25. The first payment earns interest for two periods, the second for one period, and the third earns no interest at all because it is made at the end of the annuity's life. This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

As you can see from the time line diagram, with the step-by-step approach we apply the following equation, with N = 3 and I = 5%:

$$\begin{aligned} FVA_N &= PMT(1+I)^{N-1} + PMT(1+I)^{N-2} + PMT(1+I)^{N-3} \\ &= \$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 \\ &= \$315.25 \end{aligned}$$

We can generalize and streamline the equation as follows:

$$\begin{aligned} FVA_N &= PMT(1+I)^{N-1} + PMT(1+I)^{N-2} \\ &\quad + PMT(1+I)^{N-3} + \cdots + PMT(1+I)^0 \\ &= PMT\left[\frac{(1+I)^N - 1}{I}\right] \end{aligned} \tag{2-3}$$

The first line shows the equation in its long form, and it can be transformed to the second form, which can be used to solve annuity problems with a nonfinancial calculator.[5] This equation is also built into financial calculators and

TABLE 2-3 *Summary: Future Value of an Ordinary Annuity*

	A	B	C	D	E	F	G
131	Payment amount	= PMT =	$100.00				
132	Interest rate	= I =	5.00%				
133	Number of periods	= N =	3				
134							
135		Periods:	0	1	2	3	
136							
137		Cash Flow Time Line:		−$100	−$100	−$100	
138							
139	Step-By-Step Approach.					−$100.00	
140	Multiply each payment by					−$105.00	
141	$(1+I)^{N-t}$ and sum these FVs to					−$110.25	
142	find FVA_N:					−$315.25	
143							
144	Formula Approach:						
145							
146		FVA_N =	$PMT \times \left(\frac{(1+I)^N - 1}{I}\right)$		=	$315.25	
147							
148							
149			3	5	$0	−$100.00	
150	Calculator Approach:		N	I/YR	PV	PMT	FV
151							$315.25
152							
153	Excel Function Approach:		Fixed inputs:	FVA_N =	=FV(0.05,3,−100,0) =		= $315.25
154			Cell references:	FVA_N =	=FV(C132,C133,−C131,0)=		$315.25
155		Excel entries correspond with these calculator keys:			I/YR N PMT PV		FV

[5] The long form of the equation is a geometric progression that can be reduced to the second form.

spreadsheets. With an annuity, we have recurring payments, hence the PMT key is used. Here's the calculator setup for our illustrative annuity:

3	5	0	−100		End Mode
N	I/YR	PV	PMT	FV	
				315.25	

We enter PV = 0 because we start off with nothing, and we enter PMT = −100 because we plan to deposit this amount in the account at the end of each year. When we press the FV key we get the answer, $FVA_3 = 315.25$.

Because this is an ordinary annuity, with payments coming at the *end* of each year, we must set the calculator appropriately. As noted earlier, calculators "come out of the box" set to assume that payments occur at the end of each period, that is, to deal with ordinary annuities. However, there is a key that enables us to switch between ordinary annuities and annuities due. For ordinary annuities, the designation is "End Mode" or something similar, while for annuities due the designator is "Begin" or "Begin Mode" or "Due" or something similar. If you make a mistake and set your calculator on Begin Mode when working with an ordinary annuity, then each payment would earn interest for one extra year. That would cause the compounded amounts, and thus the FVA, to be too large.

The last approach in Table 2-3 shows the spreadsheet solution, using *Excel*'s built-in function. We could put in either fixed values for N, I, and PMT or set up an Input Section, where we assign values to those variables, and then to input values into the function as cell references. Using cell references makes it easy to change the inputs to see the effects of changes on the output.

SELF TEST

For an ordinary annuity with five annual payments of $100 and a 10 percent interest rate, how many years will the first payment earn interest, and what will this payment's value be at the end? Answer this same question for the fifth payment. (4 years, $146.41; 0 years, $100)

Assume that you plan to buy a condo five years from now, and you estimate that you can save $2,500 per year. You plan to deposit the money in a bank that pays 4 percent interest, and you will make the first deposit at the end of the year. How much will you have after five years? How would your answer change if the interest rate were increased to 6 percent, or lowered to 3 percent? ($13,540.81; $14,092.73; $13,272.84)

2.8 FUTURE VALUE OF AN ANNUITY DUE

Because each payment occurs one period earlier with an annuity due, the payments will all earn interest for one additional year. Therefore, the FV of an annuity due will be greater than that of a similar ordinary annuity. If you went through the step-by-step procedure, you would see that our illustrative annuity due has an FV of $331.01 versus $315.25 for the ordinary annuity.

With the formula approach, we first use Equation 2-3, but since each payment occurs one period earlier, we multiply the Equation 2-3 result by (1 + I):

$$FVA_{due} = FVA_{ordinary}(1 + I) \tag{2-4}$$

Thus, for the annuity due, $FVA_{due} = \$315.25(1.05) = \$331.01$, which is the same result as found using the period-by-period approach. With a calculator we input the variables just as we did with the ordinary annuity, but now we set the calculator to Begin Mode to get the answer, \$331.01.

SELF TEST

Why does an annuity due always have a higher future value than an ordinary annuity?

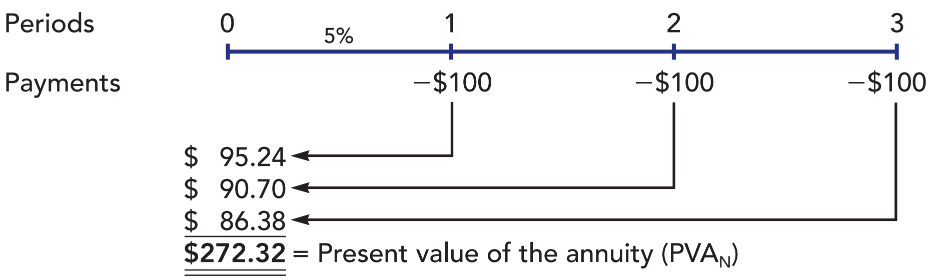
If you calculated the value of an ordinary annuity, how could you find the value of the corresponding annuity due?

Assume that you plan to buy a condo five years from now, and you need to save for a down payment. You plan to save \$2,500 per year, with the first payment made *immediately*, and you will deposit the funds in a bank account that pays 4 percent. How much will you have after five years? How much would you have if you made the deposits at the end of each year? (\$14,082.44; \$13,540.81)

2.9 PRESENT VALUE OF AN ORDINARY ANNUITY

PVA_N
The present value of an annuity of N periods.

The present value of an annuity, **PVA_N**, can be found using the step-by-step, formula, calculator, or spreadsheet methods. Look back at Table 2-3. To find the FV of the annuity, we compounded the deposits. To find the PV, we discount them, dividing each payment by (1 + I). The step-by-step procedure is diagrammed below:



Equation 2-5 expresses the step-by-step procedure in a formula. The bracketed form of the equation can be used with a scientific calculator, and it is helpful if the annuity extends out for a number of years:

$$PVA_N = PMT/(1+I)^1 + PMT/(1+I)^2 + \cdots + PMT/(1+I)^N$$

$$= PMT\left[\frac{1 - \dfrac{1}{(1+I)^N}}{I}\right] \qquad \textbf{(2-5)}$$

$$= \$100 \times [1 - 1/(1.05)^3]/0.05 = \$272.32$$

Calculators are programmed to solve Equation 2-5, so we merely input the variables and press the PV key, *being sure the calculator is set to End Mode*. The calculator setup is shown below for both an ordinary annuity and an annuity due. Note that the PV of the annuity due is larger because each payment is discounted back one less year. Note too that you could just find the PV of the ordinary annuity and then multiply by (1 + I) = 1.05, getting \$272.32(1.05) = \$285.94, the PV of the annuity due.

3	5		–100	0	End Mode
N	I/YR	PV	PMT	FV	(Ordinary Annuity)
		272.32			

3	5		–100	0	Begin Mode
N	I/YR	PV	PMT	FV	(Annuity Due)
		285.94			

SELF TEST

Why does an annuity due have a higher present value than an ordinary annuity?

If you know the present value of an ordinary annuity, how could you find the PV of the corresponding annuity due?

What is the PVA of an ordinary annuity with 10 payments of \$100 if the appropriate interest rate is 10 percent? What would PVA be if the interest rate were 4 percent? What if the interest rate were 0 percent? How would the PVA values differ if we were dealing with annuities due? (\$614.46; \$811.09; \$1,000.00; \$675.90; \$843.53; \$1,000)

Assume that you are offered an annuity that pays \$100 at the end of each year for 10 years. You could earn 8 percent on your money in other investments with equal risk. What is the most you should pay for the annuity? If the payments began immediately, how much would the annuity be worth? (\$671.01; \$724.69)

2.10 FINDING ANNUITY PAYMENTS, PERIODS, AND INTEREST RATES

We can find payments, periods, and interest rates for annuities. Here five variables come into play: N, I, PMT, FV, and PV. If we know any four, we can find the fifth.

Finding Annuity Payments, PMT

Suppose we need to accumulate \$10,000 and have it available five years from now. Suppose further that we can earn a return of 6 percent on our savings, which are currently zero. Thus, we know that FV = 10,000, PV = 0, N = 5, and I/YR = 6. We can enter these values in a financial calculator and then press the PMT key to find how large our deposits must be. The answer will, of course, depend on whether we make deposits at the end of each year (ordinary annuity) or at the beginning (annuity due). Here are the results for each type of annuity:

5	6	0		10000	End Mode
N	I/YR	PV	PMT	FV	(Ordinary Annuity)
			–1,773.96		

5	6	0		10000	Begin Mode (Annuity Due)
N	I/YR	PV	PMT	FV	
			–1,673.55		

Thus, you must save $1,773.96 per year if you make payments at the *end* of each year, but only $1,673.55 if the payments begin *immediately*. Note that the required payment for the annuity due is the ordinary annuity payment divided by (1 + I): $1,773.96/1.06 = $1,673.55. Spreadsheets can also be used to find annuity payments.

Finding the Number of Periods, N

Suppose you decide to make end-of-year deposits, but you can only save $1,200 per year. Again assuming that you would earn 6 percent, how long would it take you to reach your $10,000 goal? Here is the calculator setup:

	6	0	–1200	10000	End Mode
N	I/YR	PV	PMT	FV	
6.96					

With these smaller deposits, it would take 6.96 years to reach the $10,000 target. If you began the deposits immediately, then you would have an annuity due and N would be a bit less, 6.63 years.

Finding the Interest Rate, I

Now suppose you can only save $1,200 annually, but you still want to have the $10,000 in five years. What rate of return would enable you to achieve your goal? Here is the calculator setup:

5		0	–1200	10000	End Mode
N	I/YR	PV	PMT	FV	
	25.78				

You would need to earn a whopping 25.78 percent. About the only way to get such a high return would be to invest in speculative stocks or head to Las Vegas and the casino. Of course, speculative stocks and gambling aren't like making deposits in a bank with a guaranteed rate of return, so there's a good chance you'd end up with nothing. We'd recommend that you change your plans—save more, lower your $10,000 target, or extend your time horizon. It might be appropriate to seek a somewhat higher return, but trying to earn 25.78 percent in a 6 percent market would require taking on more risk than would be prudent.

It's easy to find rates of return with a financial calculator or a spreadsheet. However, without one of these tools you would have to go through a trial-and-error process, and that would be very time-consuming if many years were involved.

SELF TEST

Suppose you inherited $100,000 and invested it at 7 percent per year. How much could you withdraw at the *end* of each of the next 10 years? How would your answer change if you made withdrawals at the *beginning* of each year? ($14,237.75; $13,306.31)

If you had $100,000 that was invested at 7 percent and you wanted to withdraw $10,000 at the end of each year, how long would your funds last? How long would they last if you earned 0 percent? How long would they last if you earned the 7 percent but limited your withdrawal to $7,000 per year? (17.8 years; 10 years; forever)

Your rich uncle named you as the beneficiary of his life insurance policy. The insurance company gives you a choice of $100,000 today or a 12-year annuity of $12,000 at the end of each year. What rate of return is the insurance company offering? (6.11%)

Assume that you just inherited an annuity that will pay you $10,000 per year for 10 years, with the first payment being made today. A friend of your mother offers to give you $60,000 for the annuity. If you sell it, what rate of return would your mother's friend earn on his investment? If you think a "fair" return would be 6 percent, how much should you ask for the annuity? (13.70%; $78,016.92)

2.11 PERPETUITIES

Consol
A perpetual bond issued by the British government to consolidate past debts; in general, any perpetual bond.

Perpetuity
A stream of equal payments at fixed intervals expected to continue forever.

In the last section we dealt with annuities whose payments continue for a specific number of periods—for example, $100 per year for 10 years. However, some securities promise to make payments forever. For example, in 1749 the British government issued some bonds whose proceeds were used to pay off other British bonds, and since this action consolidated the government's debt, the new bonds were called **consols.** Because consols promise to pay interest forever, they are "perpetuities." The interest rate on the consols was 2.5 percent, so a bond with a face value of $1,000 would pay $25 per year in perpetuity.[6]

A **perpetuity** is simply an annuity with an extended life. Because the payments go on forever, you couldn't apply the step-by-step approach. However, it's easy to find the PV of a perpetuity with a formula found by solving Equation 2-5 with N set at infinity:[7]

$$\text{PV of a perpetuity} = \frac{\text{PMT}}{\text{I}} \tag{2-6}$$

Now we can use Equation 2-6 to find the value of a British consol with a face value of $1,000 that pays $25 per year in perpetuity. The answer depends on the interest rate. In 1888, the "going rate" as established in the financial marketplace was 2.5 percent, so at that time the consol's value was $1,000:

$$\text{Consol value}_{1888} = \$25/0.025 = \$1{,}000$$

In 2004, 116 years later, the annual payment was still $25, but the going interest rate had risen to 5.2 percent, causing the consol's value to fall to $480.77:

$$\text{Consol value}_{2004} = \$25/0.052 = \$480.77$$

Note, though, that if interest rates decline in the future, say, to 2 percent, the value of the consol will rise:

$$\text{Consol value if rates decline to 2\%} = \$25/0.02 = \$1{,}250.00$$

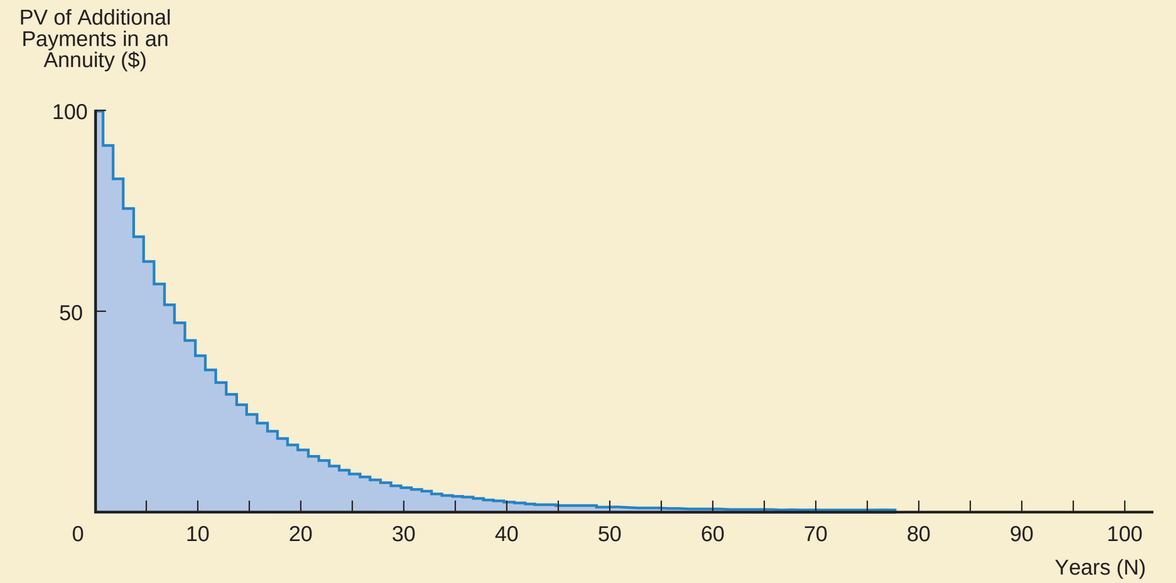
[6] The consols actually pay interest in pounds, but we discuss them in dollar terms for simplicity.
[7] Equation 2-6 was found by letting N in Equation 2-5 approach infinity. The result is Equation 2-6.

These examples demonstrate an important point: *When interest rates change, the prices of outstanding bonds also change. Bond prices decline if rates rise and increase if rates fall.* We will discuss this point in more detail in Chapter 7, where we cover bonds in depth.

Figure 2-3 gives a graphic picture of how much each payment contributes to the value of an annuity. Here we analyze an annuity that pays $100 per year when the market interest rate is 10 percent. We found the PV of each payment for the first 100 years and graphed those PVs. We also found the value of the annuity if it had a 25-year, 50-year, 100-year, and infinite life. Here are some points to note:

1. The value of an ordinary annuity is the sum of the present values of its payments.
2. We could construct graphs for annuities of any length—for 3 years, or 25 years, or 50 years, or any other period. The fewer the years, the fewer the bars in the graph.
3. As the years increase, the PV of each additional payment—which represents the amount the payment contributes to the annuity's value—decreases. This occurs because each payment is divided by $(1 + I)^t$, and that term increases exponentially with t. Indeed, in our graph the payments after 62 years are too small to be noticed.
4. The data below the graph show the value of a $100 annuity when the interest rate is 10 percent if the annuity lasts for 25, 50, and 100 years, and forever. The difference between these values shows how much the additional years contribute to the annuity's value. The payments for distant years are worth very little today, so the value of the annuity is determined largely by the

FIGURE 2-3 *Contribution of Payments to Value of $100 Annuity at 10% Interest Rate*



Value of 25-year annuity:	$907.70
Value of 50-year annuity:	$991.48
Value of 100-year annuity:	$999.93
Value of perpetuity:	$1,000.00
Amount added: Years 1–25	$907.70
26–50	$83.78
51–100	$8.45

payments to be received in the near term. Note, though, that the discount rate affects the values of distant cash flows and thus the graph. The higher the discount rate, the steeper the decline and thus the smaller the values of the distant flows.

Figure 2-3 highlights some important implications for financial issues. For example, if you win a "$10 million lottery" that actually pays $500,000 per year for 20 years, beginning immediately, the lottery is really worth a lot less than $10 million. Each cash flow must be discounted, and their sum is much less than $10 million. At a 10 percent discount rate, the "$10 million" is worth only $4,682,460, and that's before taxes. Not bad, but not $10 million.

SELF TEST

What's the present value of a perpetuity that pays $1,000 per year, beginning one year from now, if the appropriate interest rate is 5 percent? What would the value be if the annuity began its payments immediately? ($20,000, $21,000. Hint: Just add the $1,000 to be received immediately to the value of the annuity.)

Would distant payments contribute more to the value of an annuity if interest rates were high or low? (Hint: When answering conceptual questions, it often helps to make up an example and use it to help formulate your answer. PV of $100 at 5 percent after 25 years = $29.53; PV at 20 percent = $1.05. So, distant payments contribute more at low rates.)

2.12 UNEVEN CASH FLOWS

Uneven (Nonconstant) Cash Flows
A series of cash flows where the amount varies from one period to the next.

Payment (PMT)
This term designates equal cash flows coming at regular intervals.

Cash Flow (CF_t)
This term designates a cash flow that's not part of an annuity.

The definition of an annuity includes the words *constant payment*—in other words, annuities involve payments that are equal in every period. Although many financial decisions do involve constant payments, many others involve **nonconstant,** or **uneven, cash flows.** For example, the dividends on common stocks typically increase over time, and investments in capital equipment almost always generate uneven cash flows. Throughout the book, we reserve the term **payment (PMT)** for annuities with their equal payments in each period and use the term **cash flow (CF_t)** to denote uneven cash flows, where the t designates the period in which the cash flow occurs.

There are two important classes of uneven cash flows: (1) a stream that consists of a series of annuity payments plus an additional final lump sum and (2) all other uneven streams. Bonds represent the best example of the first type, while stocks and capital investments illustrate the other type. Here are numerical examples of the two types of flows:

1. Annuity plus additional final payment:

Periods	0 (I = 12%)	1	2	3	4	5
Cash flows	$0	$100	$100	$100	$100	$ 100
						$ 1,000
						$1,100

2. Irregular cash flows:

Periods	0 (I = 12%)	1	2	3	4	5
Cash flows	$0	$100	$300	$300	$300	$500

We can find the PV of either stream by using Equation 2-7 and following the step-by-step procedure, where we discount each cash flow and then sum them to find the PV of the stream:

$$PV = \frac{CF_1}{(1+I)^1} + \frac{CF_2}{(1+I)^2} + \cdots + \frac{CF_N}{(1+I)^N} = \sum_{t=1}^{N} \frac{CF_t}{(1+I)^t} \qquad \textbf{(2-7)}$$

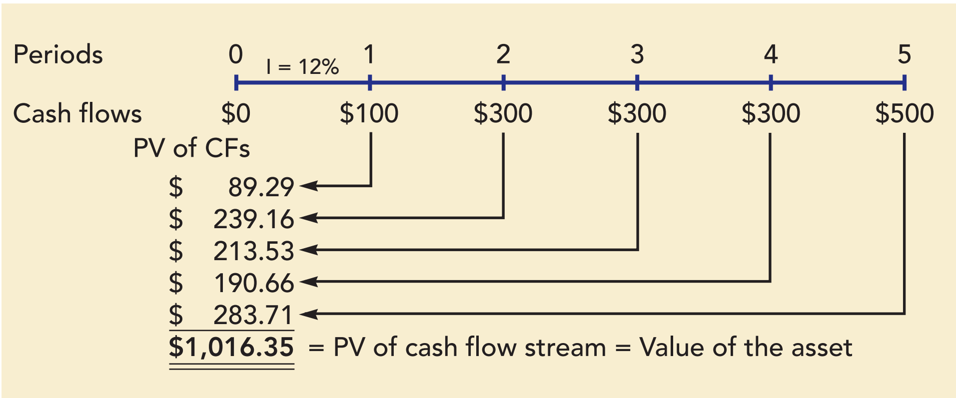
If we did this, we would find the PV of Stream 1 to be $927.90 and the PV of Stream 2 to be $1,016.35.

The step-by-step procedure is straightforward, but if we have a large number of cash flows it is time-consuming. However, financial calculators speed up the process considerably. First, consider Stream 1, and notice that here we have a five-year, 12 percent, ordinary annuity plus a final payment of $1,000. We could find the PV of the annuity, then find the PV of the final payment, and then sum them to get the PV of the stream. Financial calculators do this in one simple step—use the five TVM keys, enter the data as shown below, and then press the PV key to get the answer, $927.90.

5	12		100	1000
N	I/YR	PV	PMT	FV
		–927.90		

The solution procedure is different for the second uneven stream. Here we must use the step-by-step approach as shown in Figure 2-4. Even calculators and spreadsheets solve the problem using the step-by-step procedure, but they do it quickly and efficiently. First, you enter all the cash flows and the interest rate, then the calculator or computer discounts each cash flow to find its present value and sums these PVs to produce the PV of the stream. You must enter the cash flows in the calculator's "cash flow register," then enter the interest rate, and then press the NPV key to find the PV of the stream. NPV stands for net present value. We cover the calculator mechanics in the tutorial, and we discuss the process in more detail in Chapters 9 and 11, where we use the NPV calculation to analyze stocks and proposed capital budgeting projects. If you don't

FIGURE 2-4 *PV of an Uneven Cash Flow Stream*



know how to do the calculation with your calculator, it would be worthwhile to go to our tutorial or your calculator manual, learn the steps, and be sure you can make this calculation. You will have to learn to do it eventually, and now is a good time.

SELF TEST

Could you use Equation 2-2, once for each cash flow, to find the PV of an uneven stream of cash flows?

What's the present value of a five-year ordinary annuity of $100 plus an additional $500 at the end of Year 5 if the interest rate is 6 percent? What would the PV be if the $100 payments occurred in Years 1 through 10 and the $500 came at the end of Year 10? ($794.87; $1,015.21)

What's the present value of the following uneven cash flow stream: $0 at Time 0, $100 in Year 1 (or at Time 1), $200 in Year 2, $0 in Year 3, and $400 in Year 4 if the interest rate is 8 percent? ($558.07)

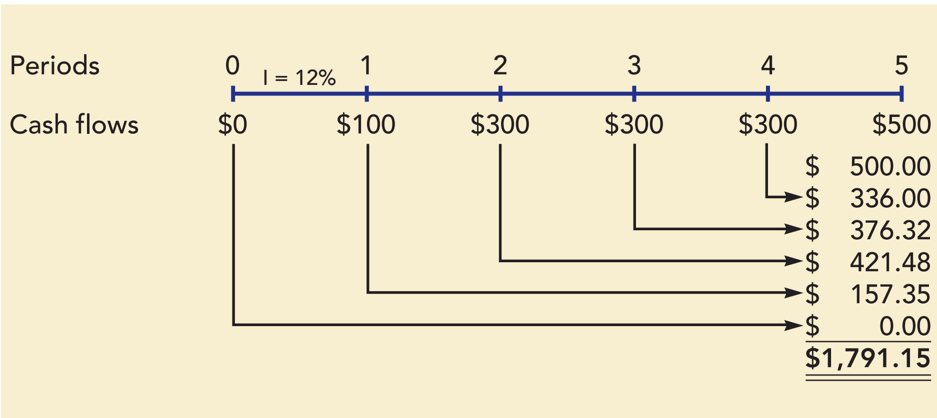
Would a typical common stock provide cash flows more like an annuity or more like an uneven cash flow stream? Explain.

2.13 FUTURE VALUE OF AN UNEVEN CASH FLOW STREAM

We find the future value of uneven cash flow streams by compounding rather than discounting. Consider Cash Flow Stream 2 in the preceding section. We discounted those cash flows to find the PV, but we would compound them to find the FV. Figure 2-5 illustrates the procedure for finding the FV of the stream, using the step-by-step approach.

The values of all financial assets—stocks, bonds, or business capital investments—are found as the present values of their expected future cash flows. Therefore, we need to calculate present values very often, far more often than future values. As a result, all financial calculators provide automated functions

FIGURE 2-5 *FV of an Uneven Cash Flow Stream*



for finding PVs, but they generally do not provide automated FV functions. On the relatively few occasions when we need to find the FV of an uneven cash flow stream, we generally use the step-by-step procedure as shown in Figure 2-5. That approach works for any cash flow stream, even those where some cash flows are zero or negative.

SELF TEST

Why are we more likely to need to calculate the PV of cash flow streams than the FV of streams?

What is the future value of this cash flow stream: $100 at the end of one year, $150 due after two years, and $300 due after three years if the appropriate interest rate is 15 percent? ($604.75)

2.14 SOLVING FOR I WITH UNEVEN CASH FLOWS[8]

Before financial calculators and spreadsheets existed, it was *extremely difficult* to find I if the cash flows were uneven. With spreadsheets and financial calculators, though, it's relatively easy to find I. If you have an annuity plus a final lump sum, you can input values for N, PV, PMT, and FV into the calculator's TVM registers and then press the I/YR key. Here is the setup for Stream 1 from Section 2.12, assuming we must pay $927.90 to buy the asset. The rate of return on the $927.90 investment is 12 percent.

5		–927.90	100	1000
N	I/YR	PV	PMT	FV
	12.00			

Finding the interest rate for an uneven cash flow stream such as Stream 2 is a bit more complicated. First, note that there is no simple procedure—finding the rate requires a trial-and-error process, which means that a financial calculator or a spreadsheet is needed. With a calculator, we would enter the CFs into the cash flow register and then press the IRR key to get the answer. IRR stands for "internal rate of return," and it is the rate of return the investment provides. The investment is the cash flow at Time 0, and it must be entered as a negative. To illustrate, consider the cash flows given below, where $CF_0 = -\$1,000$ is the cost of the asset:

Periods	0	1	2	3	4	5
Cash flows	−$1,000	$100	$300	$300	$300	$500
IRR = I =	12.55%					

When we enter those cash flows in the calculator's cash flow register and press the IRR key, we get the rate of return on the $1,000 investment, 12.55 percent. You would get the same answer using *Excel*'s IRR function. The process is covered in our calculator tutorial, and it is also discussed in Chapter 11, where we study capital budgeting.

[8] This section is relatively technical. It can be deferred at this point, but the calculations will be required in Chapter 11.

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of each of the next four years, then an extra lump sum payment of \$200 at the end of the fourth year. What is the expected rate of return on this investment? (9.05%)

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of Year 1, \$200 at the end of Year 2, and \$300 at the end of Year 3. What is the expected rate of return on this investment? (11.71%)

2.15 SEMIANNUAL AND OTHER COMPOUNDING PERIODS

Annual Compounding
The arithmetic process of determining the final value of a cash flow or series of cash flows when interest is added once a year.

Semiannual Compounding
The arithmetic process of determining the final value of a cash flow or series of cash flows when interest is added twice a year.

In all of our examples thus far we assumed that interest is compounded once a year, or annually. This is called **annual compounding.** Suppose, however, that you deposited \$100 in a bank that pays a 5 percent annual interest rate but credits interest each six months, so in the second six-month period you earn interest on your original \$100 plus interest on the interest earned during the first six months. This is called **semiannual compounding.** Note that banks generally pay interest more than once a year, virtually all bonds pay interest semiannually, and most mortgages, student loans, and auto loans require monthly payments. Therefore, it is important to understand how to deal with nonannual compounding.

To illustrate semiannual compounding, assume that we deposit \$100 in an account that pays 5 percent and leave it there for 10 years. First, consider again what the future value would be under *annual* compounding:

$$FV_N = PV(1 + I)^N = \$100(1.05)^{10} = \$162.89$$

We would, of course, get this same answer using a financial calculator or a spreadsheet.

How would things change in this example if interest were paid semiannually rather than annually? First, whenever payments occur more than once a year, you must make two conversions: (1) Convert the stated interest rate into a "periodic rate," and (2) convert the number of years into "number of periods." The conversions are done as follows, where I is the stated annual rate, M is the number of compounding periods per year, and N is the number of years:

$$\text{Periodic rate } (I_{PER}) = \text{Stated annual rate/Number of payments per year} = I/M \quad \textbf{(2-8)}$$

With a stated annual rate of 5 percent, compounded semiannually, the periodic rate is 2.5 percent:

$$\text{Periodic rate} = 5\%/2 = 2.5\%$$

The number of compounding periods per year is found with Equation 2-9:

$$\text{Number of periods} = \text{(Number of years)(Periods per year)} = NM \quad \textbf{(2-9)}$$

With 10 years and semiannual compounding, there are 20 periods:

$$\text{Number of periods} = 10(2) = 20 \text{ periods}$$

Under semiannual compounding, our \$100 investment will earn 2.5 percent every six months for 20 semiannual periods, not 5 percent per year for 10 years.

The periodic rate and number of periods, not the annual rate and number of years, must be shown on time lines and entered into the calculator or spreadsheet whenever you are working with nonannual compounding.[9]

With this background, we can find the value of $100 after 10 years if it is held in an account that pays a stated annual rate of 5.0 percent but with semiannual compounding. Here's the time line and the future value:

With a financial calculator, we would get the same result, using the periodic rate and number of periods:

The future value under semiannual compounding, $163.86, exceeds the FV under annual compounding, $162.89, because interest starts accruing sooner and thus you earn more interest on interest.

How would things change in our example if interest were compounded quarterly, or monthly, or daily? With quarterly compounding, there would be $NM = 10(4) = 40$ periods, and the periodic rate would be $I/M = 5\%/4 = 1.25\%$ per quarter. Using those values, we would find $FV = \$164.36$. If we used monthly compounding, we would have $10(12) = 120$ periods, the monthly rate would be $5\%/12 = 0.416667\%$, and the FV would rise to $164.70. If we went to daily compounding, we would have $10(365) = 3{,}650$ periods, the daily rate would be $5\%/365 = 0.0136986\%$ per day, and the FV would be $164.87, based on a 365-day year.

The same logic applies when we find present values under semiannual compounding. Again, we use Equation 2-8 to convert the stated annual rate to the periodic (semiannual) rate and Equation 2-9 to find the number of semiannual periods. We then use the periodic rate and number of periods in the calculations. For example, we can find the PV of $100 due after 10 years when the stated annual rate is 5 percent, with semiannual compounding:

$$\text{Periodic rate} = 5\%/2 = 2.5\% \text{ per period}$$

$$\text{Number of periods} = 10(2) = 20 \text{ periods}$$

$$\text{PV of } \$100 = \$100/(1.025)^{20} = \$61.03$$

We would get this same result with a financial calculator:

[9] With some financial calculators, you can enter the annual (nominal) rate and the number of compounding periods per year rather than make the conversions we recommend. We prefer the conversions because they must be used on time lines and also because it is easy to forget to reset your calculator after you change its settings, which may lead to an error on your next problem.

If we increased the number of compounding periods from 2 (semiannual) to 12 (monthly), the PV would decline to $60.72, and if we went to daily compounding, it would fall to $60.66.

SELF TEST

Would you rather invest in an account that pays 7 percent with annual compounding or 7 percent with monthly compounding? Would you rather borrow at 7 percent and make annual or monthly payments? Why?

What's the *future value* of $100 after three years if the appropriate interest rate is 8 percent, compounded annually? Compounded monthly? ($125.97; $127.02)

What's the *present value* of $100 due in three years if the appropriate interest rate is 8 percent, compounded annually? Compounded monthly? ($79.38; $78.73)

2.16 COMPARING INTEREST RATES

Different compounding periods are used for different types of investments. For example, bank accounts generally pay interest daily; most bonds pay interest semiannually; stocks pay dividends quarterly; and mortgages, auto loans, and other instruments require monthly payments.[10] If we are to compare investments or loans with different compounding periods properly, we need to put them on a common basis. Here are some terms you need to understand:

Nominal (Quoted, or Stated) Interest Rate, I_{NOM}
The contracted, or quoted, or stated, interest rate.

Annual Percentage Rate (APR)
The periodic rate times the number of periods per year.

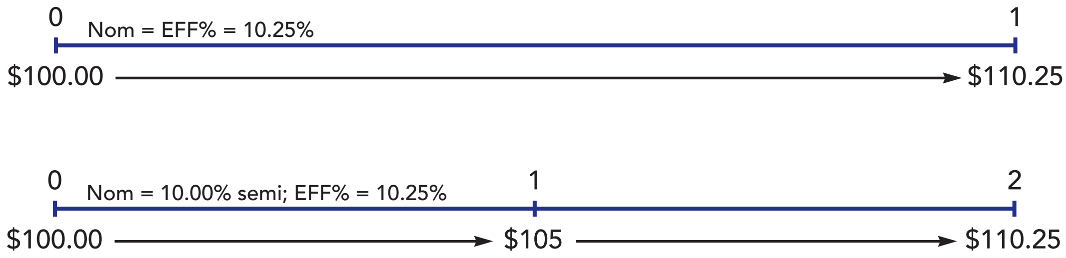
Effective (Equivalent) Annual Rate (EFF% or EAR)
The annual rate of interest actually being earned, as opposed to the quoted rate. Also called the "equivalent annual rate."

- The **nominal rate (I_{NOM}),** also called the **annual percentage rate** (or **APR**), or **stated,** or **quoted rate,** is the rate that banks, credit card companies, student loan officers, auto dealers, and so on, tell you they are charging on loans or paying on deposits. Note that if two banks offer loans with a stated rate of 8 percent but one requires monthly payments and the other quarterly payments, then they are not charging the same "true" rate—the one that requires monthly payments is really charging more than the one with quarterly payments because it will get your money sooner. So, to compare loans across lenders, or interest rates earned on different securities, you should calculate effective annual rates as described here.[11]
- The **effective annual rate,** abbreviated **EFF%,** is also called the **equivalent annual rate (EAR).** This is the rate that would produce the same future value under annual compounding as would more frequent compounding at a given nominal rate.
- If a loan or investment uses annual compounding, then its nominal rate is also its effective rate. However, if compounding occurs more than once a year, the EFF% is higher than I_{NOM}.
- To illustrate, a nominal rate of 10 percent, with semiannual compounding, is equivalent to a rate of 10.25 percent with annual compounding because both of those rates will cause $100 to grow to the same amount after one year.

[10] Some banks even pay interest compounded *continuously*. Continuous compounding is discussed in Web Appendix 2A.

[11] Note, though, if you are comparing two bonds that both pay interest semiannually, then it's OK to compare their nominal rates. Similarly, you can compare the nominal rates on two money funds that pay interest daily. But don't compare the nominal rate on a semiannual bond with the nominal rate on a money fund that compounds daily, because that will make the money fund look worse than it really is.

The top line in the following diagram shows that \$100 will grow to \$110.25 at a nominal rate of 10.25 percent. The lower line shows the situation if the nominal rate is 10 percent but semiannual compounding is used.



We can find the effective annual rate, given the nominal rate and the number of compounding periods per year, with this equation:

$$\text{Effective annual rate (EFF\%)} = \left(1 + \frac{I_{NOM}}{M}\right)^M - 1.0 \qquad \textbf{(2-10)}$$

Here I_{NOM} is the nominal rate expressed as a decimal and M is the number of compounding periods per year. In our example, the nominal rate is 10 percent but with semiannual compounding, hence $I_{NOM} = 10\% = 0.10$ and $M = 2$. This results in EFF% = 10.25%:[12]

$$\text{Effective annual rate (EFF\%)} = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 0.1025 = 10.25\%$$

Thus, if one investment promises to pay 10 percent with semiannual compounding and an equally risky investment promises 10.25 percent with annual compounding, we would be indifferent between the two.

SELF TEST

Define the terms "annual percentage rate, or APR," "effective annual rate, or EFF%," and "nominal interest rate, I_{NOM}."

A bank pays 5 percent with daily compounding on its savings accounts. Should it advertise the nominal or effective rate if it is seeking to attract new deposits?

Credit card issuers must by law print their annual percentage rate on their monthly statements. A common APR is 18 percent, with interest paid monthly. What is the EFF% on such a loan? [EFF% = $(1 + 0.18/12)^{12} - 1 = 0.1956 = 19.56\%$.]

Some years ago banks didn't have to reveal the rate they charged on credit cards. Then Congress passed a "truth in lending" law that required them to publish their APR. Is the APR really the "most truthful" rate, or would the EFF% be "more truthful"?

[12] Most financial calculators are programmed to find the EFF% or, given the EFF%, to find the nominal rate. This is called "interest rate conversion." You enter the nominal rate and the number of compounding periods per year and then press the EFF% key to find the effective annual rate. However, we generally use Equation 2-10 because it's as easy to use as the interest conversion feature is, and the equation reminds us of what we are really doing. If you use the interest rate conversion feature on your calculator, don't forget to reset your calculator settings. Interest conversion is discussed in the calculator tutorials.

2.17 FRACTIONAL TIME PERIODS

Thus far we have assumed that payments occur at either the beginning or the end of periods, but not *within* periods. However, we often encounter situations that require compounding or discounting over fractional periods. For example, suppose you deposited $100 in a bank that pays a nominal rate of 10 percent but adds interest daily, based on a 365-day year. How much would you have after 9 months? The answer is $107.79, found as follows:[13]

$$\text{Periodic rate} = I_{PER} = 0.10/365 = 0.000273973 \text{ per day}$$
$$\text{Number of days} = (9/12)(365) = 0.75(365) = 273.75 \text{ rounded to } 274$$
$$\text{Ending amount} = \$100(1.000273973)^{274} = \$107.79$$

Now suppose you borrow $100 from a bank whose nominal rate is 10 percent per year "simple interest," which means that interest is not earned on interest. If the loan is outstanding for 274 days, how much interest would you have to pay? Here we would calculate a daily interest rate, I_{PER}, as above, but multiply it by 274 rather than use the 274 as an exponent:

$$\text{Interest owed} = \$100(0.000273973)(274) = \$7.51$$

You would owe the bank a total of $107.51 after 274 days. This is the procedure most banks actually use to calculate interest on loans, except that they require borrowers to pay the interest on a monthly basis rather than after 274 days.

SELF TEST

Suppose a company borrowed $1 million at a rate of 9 percent, simple interest, with interest paid at the end of each month. The bank uses a 360-day year. How much interest would the firm have to pay in a 30-day month? What would the interest be if the bank used a 365-day year? [(0.09/360)(30)($1,000,000) = $7,500 interest for the month. For the 365-day year, (0.09/365)(30)($1,000,000) = $7,397.26 of interest. The use of a 360-day year raises the interest cost by $102.74. That's why banks like to use it on loans.]

Suppose you deposited $1,000 in a credit union that pays 7 percent with daily compounding and a 365-day year. What is the EFF%, and how much could you withdraw after seven months, assuming this is seven-twelfths of a year? [EFF% = $(1 + 0.07/365)^{365} - 1$ = 0.07250098 = 7.250098%. Thus, your account would grow from $1,000 to $\$1{,}000(1.07250098)^{0.583333}$ = $1,041.67, and you could withdraw that amount.]

2.18 AMORTIZED LOANS[14]

Amortized Loan
A loan that is repaid in equal payments over its life.

An important application of compound interest involves loans that are paid off in installments over time. Included are automobile loans, home mortgage loans, student loans, and many business loans. A loan that is to be repaid in equal amounts on a monthly, quarterly, or annual basis is called an **amortized loan.**[15]

---

[13] Bank loan contracts specifically state whether they are based on a 360- or a 365-day year. If a 360-day year is used, then the daily rate is higher, so the effective rate is also higher. Here we assumed a 365-day year. Also, note that in real-world calculations, banks' computers have built-in calendars, so they can calculate the exact number of days, taking account of 30-day, 31-day, and 28- or 29-day months.

[14] Amortized loans are important, but this section can be omitted without loss of continuity.

[15] The word *amortized* comes from the Latin *mors*, meaning "death," so an amortized loan is one that is "killed off" over time.

Table 2-4 illustrates the amortization process. A homeowner borrows \$100,000 on a mortgage loan, and the loan is to be repaid in five equal payments at the end of each of the next five years.[16] The lender charges 6 percent on the balance at the beginning of each year.

Our first task is to determine the payment the homeowner must make each year. Here's a picture of the situation:

The payments must be such that the sum of their PVs equals \$100,000:

$$\$100{,}000 = \frac{PMT}{(1.06)^1} + \frac{PMT}{(1.06)^2} + \frac{PMT}{(1.06)^3} + \frac{PMT}{(1.06)^4} + \frac{PMT}{(1.06)^5} = \sum_{t=1}^{5} \frac{PMT}{(1.06)^t}$$

We could insert values into a calculator as shown below to get the required payments, \$23,739.64:[17]

Therefore, the borrower must pay the lender \$23,739.64 per year for the next five years.

TABLE 2-4 *Loan Amortization Schedule, \$100,000 at 6% for 5 Years*

Amount borrowed: \$100,000
Years: 5
Rate: 6%
PMT: −\$23,739.64

Year	Beginning Amount (1)	Payment (2)	Interest[a] (3)	Repayment of Principal[b] (4)	Ending Balance (5)
1	\$100,000.00	\$23,739.64	\$6,000.00	\$17,739.64	\$82,260.36
2	82,260.36	23,739.64	4,935.62	18,804.02	63,456.34
3	63,456.34	23,739.64	3,807.38	19,932.26	43,524.08
4	43,524.08	23,739.64	2,611.44	21,128.20	22,395.89
5	22,395.89	23,739.64	1,343.75	22,395.89	0.00

[a] Interest in each period is calculated by multiplying the loan balance at the beginning of the year by the interest rate. Therefore, interest in Year 1 is \$100,000.00(0.6) = \$6,000; in Year 2 it is \$4,935.62; and so on.
[b] Repayment of principal is equal to the payment of \$23,739.64 minus the interest charge for the year.

[16] Most mortgage loans call for monthly payments over 10 to 30 years, but we use a shorter period to reduce the calculations.
[17] You could also factor out the PMT term, find the value of the remaining summation term (it's 4.212364), and then divide it into the \$100,000 to find the payment, \$23,739.64.

Amortization Schedule
A table showing precisely how a loan will be repaid. It gives the required payment on each payment date and a breakdown of the payment, showing how much is interest and how much is repayment of principal.

Each payment will consist of two parts—interest and repayment of principal. This breakdown is shown on an **amortization schedule** such as the one in Table 2-4. The interest component is relatively high in the first year, but it declines as the loan balance decreases. For tax purposes, the borrower would deduct the interest component while the lender would report the same amount as taxable income.

SELF TEST

Suppose you borrowed $30,000 on a student loan at a rate of 8 percent and now must repay it in three equal installments at the end of each of the next three years. How large would your payments be, how much of the first payment would represent interest, how much would be principal, and what would your ending balance be after the first year? (PMT = $11,641.01; Interest = $2,400; Principal = $9,241.01; Balance at end of Year 1 = $20,758.99)

Tying It All Together

In this chapter we worked with single payments, ordinary annuities, annuities due, perpetuities, and uneven cash flow streams. There is one fundamental equation, Equation 2-1, which is used to calculate the future value of a given amount. The equation can be transformed to Equation 2-2 and then used to find the present value of a given future amount. We used time lines to show when cash flows occur, and we saw that time value problems can be solved in a step-by-step manner where we work with individual cash flows, with formulas that streamline the approach, with financial calculators, and with spreadsheets.

As we noted at the outset, TVM is the single most important concept in finance, and the procedures developed in Chapter 2 are used throughout the book. Time value analysis is used to find the values of stocks, bonds, and capital budgeting projects. It is also used to analyze personal finance problems, like the retirement issue set forth in the opening vignette. You will become more familiar with time value analysis as you go through the book, but we *strongly recommend* that you get a good handle on Chapter 2 before you continue.

SELF-TEST QUESTIONS AND PROBLEMS (Solutions Appear in Appendix A)

ST-1 **Key terms** Define each of the following terms:

a. Time line
b. FV_N; PV; I; INT; N; FVA_N; PMT; PVA_N
c. Compounding; discounting
d. Simple interest; compound interest

e. Opportunity cost
f. Annuity; ordinary (deferred) annuity; annuity due
g. Consol; perpetuity
h. Uneven cash flow; payment; cash flow (CF_t)
i. Annual compounding; semiannual compounding
j. Nominal (quoted) interest rate; annual percentage rate (APR); effective (equivalent) annual rate (EAR or EFF%)
k. Amortized loan; amortization schedule

ST-2 **Future value** It is now January 1, 2006. You will deposit $1,000 today into a savings account that pays 8 percent.

a. If the bank compounds interest annually, how much will you have in your account on January 1, 2009?
b. What would your January 1, 2009, balance be if the bank used quarterly compounding?
c. Suppose you deposit $1,000 in 3 payments of $333.333 each on January 1 of 2007, 2008, and 2009. How much would you have in your account on January 1, 2009, based on 8 percent annual compounding?
d. How much would be in your account if the 3 payments began on January 1, 2006?
e. Suppose you deposit 3 equal payments in your account on January 1 of 2007, 2008, and 2009. Assuming an 8 percent interest rate, how large must your payments be to have the same ending balance as in part a?

ST-3 **Time value of money** It is now January 1, 2006, and you will need $1,000 on January 1, 2010, in 4 years. Your bank compounds interest at an 8 percent annual rate.

a. How much must you deposit today to have a balance of $1,000 on January 1, 2010?
b. If you want to make 4 equal payments on each January 1 from 2007 through 2010 to accumulate the $1,000, how large must each payment be? (Note that the payments begin a year from today.)
c. If your father were to offer either to make the payments calculated in part b ($221.92) or to give you $750 on January 1, 2007 (a year from today), which would you choose? Explain.
d. If you have only $750 on January 1, 2007, what interest rate, compounded annually for 3 years, must you earn to have $1,000 on January 1, 2010?
e. Suppose you can deposit only $200 each January 1 from 2007 through 2010 (4 years). What interest rate, with annual compounding, must you earn to end up with $1,000 on January 1, 2010?
f. Your father offers to give you $400 on January 1, 2007. You will then make 6 additional equal payments each 6 months from July 2007 through January 2010. If your bank pays 8 percent, compounded semiannually, how large must each payment be for you to end up with $1,000 on January 1, 2010?
g. What is the EAR, or EFF%, earned on the bank account in part f? What is the APR earned on the account?

ST-4 **Effective annual rates** Bank A offers loans at an 8 percent nominal rate (its APR), but requires that interest be paid quarterly; that is, it uses quarterly compounding. Bank B wants to charge the same effective rate on its loans, but it wants to collect interest on a monthly basis, that is, use monthly compounding. What nominal rate must Bank B set?

QUESTIONS

2-1 What is an *opportunity cost*? How is this concept used in TVM analysis, and where is it shown on a time line? Is a single number used in all situations? Explain.

2-2 Explain whether the following statement is true or false: $100 a year for 10 years is an annuity, but $100 in Year 1, $200 in Year 2, and $400 in Years 3 through 10 does *not* constitute an annuity. However, the second series *contains* an annuity.

2-3 If a firm's earnings per share grew from $1 to $2 over a 10-year period, the *total growth* would be 100 percent, but the *annual growth rate* would be *less than* 10 percent. True or false? Explain. (Hint: If you aren't sure, plug in some numbers and check it out.)

2-4 Would you rather have a savings account that pays 5 percent interest compounded semiannually or one that pays 5 percent interest compounded daily? Explain.

2-5 To find the present value of an uneven series of cash flows, you must find the PVs of the individual cash flows and then sum them. Annuity procedures can never be of use, even if some of the cash flows constitute an annuity because the entire series is not an annuity. True or false? Explain.

2-6 The present value of a perpetuity is equal to the payment on the annuity, PMT, divided by the interest rate, I: PV = PMT/I. What is the *future value* of a perpetuity of PMT dollars per year? (Hint: The answer is infinity, but explain why.)

2-7 Banks and other lenders are required to disclose a rate called the APR. What is this rate? Why did Congress require that it be disclosed? Is it the same as the effective annual rate? If you were comparing the costs of loans from different lenders, could you use their APRs to determine the one with the lowest effective interest rate? Explain.

2-8 What is a loan amortization schedule, and what are some ways these schedules are used?

PROBLEMS

Easy Problems 1–8

2-1 **Future value** If you deposit $10,000 in a bank account that pays 10 percent interest annually, how much would be in your account after 5 years?

2-2 **Present value** What is the present value of a security that will pay $5,000 in 20 years if securities of equal risk pay 7 percent annually?

2-3 **Finding the required interest rate** Your parents will retire in 18 years. They currently have $250,000, and they think they will need $1,000,000 at retirement. What annual interest rate must they earn to reach their goal, assuming they don't save any additional funds?

2-4 **Time for a lump sum to double** If you deposit money today in an account that pays 6.5 percent annual interest, how long will it take to double your money?

2-5 **Time to reach a financial goal** You have $42,180.53 in a brokerage account, and you plan to deposit an additional $5,000 at the end of every future year until your account totals $250,000. You expect to earn 12 percent annually on the account. How many years will it take to reach your goal?

2-6 **Future value: annuity versus annuity due** What's the future value of a 7 percent, 5-year ordinary annuity that pays $300 each year? If this were an annuity due, what would its future value be?

2-7 **Present and future values of a cash flow stream** An investment will pay $100 at the end of each of the next 3 years, $200 at the end of Year 4, $300 at the end of Year 5, and $500 at the end of Year 6. If other investments of equal risk earn 8 percent annually, what is its present value? Its future value?

2-8 **Loan amortization and EAR** You want to buy a car, and a local bank will lend you $20,000. The loan would be fully amortized over 5 years (60 months), and the nominal interest rate would be 12 percent, with interest paid monthly. What would be the monthly loan payment? What would be the loan's EAR?

Intermediate Problems 9–26

2-9 **Present and future values for different periods** Find the following values, *using the equations* and then a financial calculator. Compounding/discounting occurs annually.

a. An initial $500 compounded for 1 year at 6 percent.
b. An initial $500 compounded for 2 years at 6 percent.
c. The present value of $500 due in 1 year at a discount rate of 6 percent.
d. The present value of $500 due in 2 years at a discount rate of 6 percent.

2-10 **Present and future values for different interest rates** Find the following values. Compounding/discounting occurs annually.

a. An initial $500 compounded for 10 years at 6 percent.
b. An initial $500 compounded for 10 years at 12 percent.
c. The present value of $500 due in 10 years at 6 percent.
d. The present value of $1,552.90 due in 10 years at 12 percent and also at 6 percent.
e. Define *present value*, and illustrate it using a time line with data from part d. How are present values affected by interest rates?

2-11 **Growth rates** Shalit Corporation's 2005 sales were $12 million. Its 2000 sales were $6 million.

a. At what rate have sales been growing?
b. Suppose someone made this statement: "Sales doubled in 5 years. This represents a growth of 100 percent in 5 years, so, dividing 100 percent by 5, we find the growth rate to be 20 percent per year." Is the statement correct?

2-12 **Effective rate of interest** Find the interest rates earned on each of the following:

a. You *borrow* $700 and promise to pay back $749 at the end of 1 year.
b. You *lend* $700 and the borrower promises to pay you $749 at the end of 1 year.
c. You *borrow* $85,000 and promise to pay back $201,229 at the end of 10 years.
d. You *borrow* $9,000 and promise to make payments of $2,684.80 at the end of each year for 5 years.

2-13 **Time for a lump sum to double** How long will it take $200 to double if it earns the following rates? Compounding occurs once a year.

a. 7 percent.
b. 10 percent.
c. 18 percent.
d. 100 percent.

2-14 **Future value of an annuity** Find the *future values* of these *ordinary annuities*. Compounding occurs once a year.

a. $400 per year for 10 years at 10 percent.
b. $200 per year for 5 years at 5 percent.
c. $400 per year for 5 years at 0 percent.
d. Rework parts a, b, and c assuming that they are *annuities due*.

2-15 **Present value of an annuity** Find the *present values* of these *ordinary annuities*. Discounting occurs once a year.

a. $400 per year for 10 years at 10 percent.
b. $200 per year for 5 years at 5 percent.
c. $400 per year for 5 years at 0 percent.
d. Rework parts a, b, and c assuming that they are *annuities due*.

2-16 **Present value of a perpetuity** What is the present value of a $100 perpetuity if the interest rate is 7 percent? If interest rates doubled to 14 percent, what would its present value be?

2-17 **Effective interest rate** You borrow $85,000; the annual loan payments are $8,273.59 for 30 years. What interest rate are you being charged?

2-18 **Uneven cash flow stream**

a. Find the present values of the following cash flow streams at 8 percent, compounded annually.

	0	1	2	3	4	5
Stream A	$0	$100	$400	$400	$400	$300
Stream B	$0	$300	$400	$400	$400	$100

b. What are the PVs of the streams at 0 percent, compounded annually?

2-19 **Future value of an annuity** Your client is 40 years old, and she wants to begin saving for retirement, with the first payment to come one year from now. She can save $5,000 per year, and you advise her to invest it in the stock market, which you expect to provide an average return of 9 percent in the future.

a. If she follows your advice, how much money would she have at 65?
b. How much would she have at 70?
c. If she expects to live for 20 years in retirement if she retires at 65 and for 15 years at 70, and her investments continue to earn the same rate, how much could she withdraw at the end of each year after retirement at each retirement age?

2-20 **PV of a cash flow stream** A rookie quarterback is negotiating his first NFL contract. His opportunity cost is 10 percent. He has been offered three possible 4-year contracts.

Payments are guaranteed, and they would be made at the end of each year. Terms of each contract are listed below:

	1	2	3	4
Contract 1	$3,000,000	$3,000,000	$3,000,000	$3,000,000
Contract 2	$2,000,000	$3,000,000	$4,000,000	$5,000,000
Contract 3	$7,000,000	$1,000,000	$1,000,000	$1,000,000

As his advisor, which would you recommend that he accept?

2-21 **Evaluating lump sums and annuities** Crissie just won the lottery, and she must choose between three award options. She can elect to receive a lump sum today of $61 million, to receive 10 end-of-year payments of $9.5 million, or 30 end-of-year payments of $5.5 million.

a. If she thinks she can earn 7 percent annually, which should she choose?
b. If she expects to earn 8 percent annually, which is the best choice?
c. If she expects to earn 9 percent annually, which would you recommend?
d. Explain how interest rates influence the optimal choice.

2-22 **Loan amortization** Jan sold her house on December 31 and took a $10,000 mortgage as part of the payment. The 10-year mortgage has a 10 percent nominal interest rate, but it calls for semiannual payments beginning next June 30. Next year, Jan must report on Schedule B of her IRS Form 1040 the amount of interest that was included in the 2 payments she received during the year.

a. What is the dollar amount of each payment Jan receives?
b. How much interest was included in the first payment? How much repayment of principal? How do these values change for the second payment?
c. How much interest must Jan report on Schedule B for the first year? Will her interest income be the same next year?
d. If the payments are constant, why does the amount of interest income change over time?

2-23 **Future value for various compounding periods** Find the amount to which $500 will grow under each of these conditions:

a. 12 percent compounded annually for 5 years.
b. 12 percent compounded semiannually for 5 years.
c. 12 percent compounded quarterly for 5 years.
d. 12 percent compounded monthly for 5 years.
e. 12 percent compounded daily for 5 years.
f. Why does the observed pattern of FVs occur?

2-24 **Present value for various compounding periods** Find the present value of $500 due in the future under each of these conditions:

a. 12 percent nominal rate, semiannual compounding, discounted back 5 years.
b. 12 percent nominal rate, quarterly compounding, discounted back 5 years.
c. 12 percent nominal rate, monthly compounding, discounted back 1 year.
d. Why do the differences in the PVs occur?

2-25 **Future value of an annuity** Find the future values of the following ordinary annuities:

a. FV of $400 paid each 6 months for 5 years at a nominal rate of 12 percent, compounded semiannually.
b. FV of $200 paid each 3 months for 5 years at a nominal rate of 12 percent, compounded quarterly.
c. These annuities receive the same amount of cash during the 5-year period and earn interest at the same nominal rate, yet the annuity in part b ends up larger than the one in part a. Why does this occur?

2-26 **PV and loan eligibility** You have saved $4,000 for a down payment on a new car. The largest monthly payment you can afford is $350. The loan would have a 12 percent APR based on end-of-month payments. What is the most expensive car you could afford if you finance it for 48 months? For 60 months?

Challenging Problems 27–40

2-27 **Effective versus nominal interest rates** Bank A pays 4 percent interest, compounded annually, on deposits, while Bank B pays 3.5 percent, compounded daily.

a. Based on the EAR (or EFF%), which bank should you use?
b. Could your choice of banks be influenced by the fact that you might want to withdraw your funds during the year as opposed to at the end of the year? Assume that your funds must be left on deposit during an entire compounding period in order to receive any interest.

2-28 **Nominal interest rate and extending credit** As a jewelry store manager, you want to offer credit, with interest on outstanding balances paid monthly. To carry receivables, you must borrow funds from your bank at a nominal 6 percent, monthly compounding. To offset your overhead, you want to charge your customers an EAR (or EFF%) that is 2 percent more than the bank is charging you. What APR rate should you charge your customers?

2-29 **Building credit cost into prices** Your firm sells for cash only, but it is thinking of offering credit, allowing customers 90 days to pay. Customers understand the time value of money, so they would all wait and pay on the 90th day. To carry these receivables, you would have to borrow funds from your bank at a nominal 12 percent, daily compounding based on a 360-day year. You want to increase your base prices by exactly enough to offset your bank interest cost. To the closest whole percentage point, by how much should you raise your product prices?

2-30 **Reaching a financial goal** Erika and Kitty, who are twins, just received $30,000 each for their 25th birthdays. They both have aspirations to become millionaires. Each plans to make a $5,000 annual contribution to her "early retirement fund" on her birthday, beginning a year from today. Erika opened an account with the Safety First Bond Fund, a mutual fund that invests in high-quality bonds whose investors have earned 6 percent per year in the past. Kitty invested in the New Issue Bio-Tech Fund, which invests in small, newly issued bio-tech stocks and whose investors on average have earned 20 percent per year in the fund's relatively short history.

a. If the two women's funds earn the same returns in the future as in the past, how old will each be when she becomes a millionaire?
b. How large would Erika's annual contributions have to be for her to become a millionaire at the same age as Kitty, assuming their expected returns are realized?
c. Is it rational or irrational for Erika to invest in the bond fund rather than in stocks?

2-31 **Required lump sum payment** You need $10,000 annually for 4 years to complete your education, starting next year. (One year from today you would withdraw the first $10,000.) Your uncle will deposit an amount *today* in a bank paying 5 percent annual interest, which would provide the needed $10,000 payments.

a. How large must the deposit be?
b. How much will be in the account immediately after you make the first withdrawal?

2-32 **Reaching a financial goal** Six years from today you need $10,000. You plan to deposit $1,500 annually, with the first payment to be made a year from today, in an account that pays an 8 percent effective annual rate. Your last deposit will be for less than $1,500 if less is needed to have the $10,000 in 6 years. How large will your last payment be?

2-33 **FV of uneven cash flow** You want to buy a house within 3 years, and you are currently saving for the down payment. You plan to save $5,000 at the end of the first year, and you anticipate that your annual savings will increase by 10 percent annually thereafter. Your expected annual return is 7 percent. How much would you have for a down payment at the end of Year 3?

2-34 **Amortization schedule**

a. Set up an amortization schedule for a $25,000 loan to be repaid in equal installments at the end of each of the next 3 years. The interest rate is 10 percent, compounded annually.
b. What percentage of the payment represents interest and what percentage represents principal for each of the 3 years? Why do these percentages change over time?

2-35 **Amortization schedule with a balloon payment** You want to buy a house that costs $100,000. You have $10,000 for a down payment, but your credit is such that mortgage companies will not lend you the required $90,000. However, the realtor persuades the seller to take a $90,000 mortgage (called a seller take-back mortgage) at a rate of 7 percent, provided the loan is paid off in full in 3 years. You expect to inherit $100,000 in 3 years, but right now all you have is $10,000, and you can only afford to make payments of no more than $7,500 per year given your salary. (The loan would really call for monthly payments, but assume end-of-year annual payments to simplify things.)

a. If the loan were amortized over 3 years, how large would each annual payment be? Could you afford those payments?
b. If the loan were amortized over 30 years, what would each payment be, and could you afford those payments?
c. To satisfy the seller, the 30-year mortgage loan would be written as a "balloon note," which means that at the end of the 3rd year you would have to make the regular payment plus the remaining balance on the loan. What would the loan balance be at the end of Year 3, and what would the balloon payment be?

2-36 **Nonannual compounding**

a. You plan to make 5 deposits of $1,000 each, one every 6 months, with the first payment being made in 6 months. You will then make no more deposits. If the bank pays 4 percent nominal interest, compounded semiannually, how much would be in your account after 3 years?
b. One year from today you must make a payment of $10,000. To prepare for this payment, you plan to make 2 equal quarterly deposits, in 3 and 6 months, in a bank that pays 4 percent nominal interest, compounded quarterly. How large must each of the 2 payments be?

2-37 **Paying off credit cards** Simon recently received a credit card with an 18 percent nominal interest rate. With the card, he purchased a new stereo for $350.00. The minimum payment on the card is only $10 per month.

a. If he makes the minimum monthly payment and makes no other charges, how long will it be before he pays off the card? Round to the nearest month.
b. If he makes monthly payments of $30, how long will it take him to pay off the debt? Round to the nearest month.
c. How much more in total payments will he make under the $10-a-month plan than under the $30-a-month plan?

2-38 **PV and a lawsuit settlement** It is now December 31, 2005, and a jury just found in favor of a woman who sued the city for injuries sustained in a January 2004 accident. She requested recovery of lost wages, plus $100,000 for pain and suffering, plus $20,000 for her legal expenses. Her doctor testified that she has been unable to work since the accident and that she will not be able to work in the future. She is now 62, and the jury decided that she would have worked for another 3 years. She was scheduled to have earned $34,000 in 2004, and her employer testified that she would probably have received raises of 3 percent per year. The actual payment will be made on December 31, 2006. The judge stipulated that all dollar amounts are to be adjusted to a present value basis on December 31, 2006, using a 7 percent annual interest rate, using compound, not simple, interest. Furthermore, he stipulated that the pain and suffering and legal expenses should be based on a December, 31, 2005, date. How large a check must the city write on December 31, 2006?

2-39 **Required annuity payments** Your father is 50 years old and will retire in 10 years. He expects to live for 25 years after he retires, until he is 85. He wants a fixed retirement income that has the same purchasing power at the time he retires as $40,000 has today. (The real value of his retirement income will decline annually after he retires.) His *retirement income will begin the day he retires*, 10 years from today; and he will then receive 24 additional annual payments. Annual inflation is expected to be 5 percent. He currently has $100,000 saved, and he expects to earn 8 percent annually on his savings. How much must he save during each of the next 10 years (end-of-year deposits) to meet his retirement goal?

2-40 **Required annuity payments** A father is now planning a savings program to put his daughter through college. She is 13, she plans to enroll at the university in 5 years, and she should graduate in 4 years. Currently, the annual cost (for everything—food, clothing, tuition, books, transportation, and so forth) is $15,000, but these costs are expected to increase by 5 percent annually. The college requires that this amount be paid at the start of the year. She now has $7,500 in a college savings account that pays 6 percent annually. The father will make 6 equal annual deposits into her account; the 1st deposit today and the 6th on the day she starts college. How large must each of the 6 payments be? [Hint: Calculate the cost (inflated at 5 percent) for each year of college, then find the total present value of those costs, discounted at 6 percent, as of the day she enters college. Then find the compounded value of her initial $7,500 on that same day. The difference between the PV costs and the amount that would be in the savings account must be made up by the father's deposits, so find the 6 equal payments (starting immediately) that will compound to the required amount.]

COMPREHENSIVE/SPREADSHEET PROBLEM

2-41 **Time value of money** Answer the following questions:

a. Find the FV of $1,000 after 5 years earning a rate of 10 percent annually.
b. What would the investment's FV be at rates of 0 percent, 5 percent, and 20 percent after 0, 1, 2, 3, 4, and 5 years?
c. Find the PV of $1,000 due in 5 years if the discount rate is 10 percent.
d. What is the rate of return on a security that costs $1,000 and returns $2,000 after 5 years?
e. Suppose California's population is 30 million people, and its population is expected to grow by 2 percent annually. How long would it take for the population to double?
f. Find the PV of an ordinary annuity that pays $1,000 each of the next 5 years if the interest rate is 15 percent. What is the annuity's FV?
g. How would the PV and FV of the above annuity change if it were an annuity due?
h. What would the FV and the PV be for $1,000 due in 5 years if the interest rate were 10 percent, semiannual compounding?
i. What would the annual payments be for an ordinary annuity for 10 years with a PV of $1,000 if the interest rate were 8 percent? What would the payments be if this were an annuity due?
j. Find the PV and the FV of an investment that pays 8 percent annually and makes the following end-of-year payments:

0	1	2	3
	$100	$200	$400

k. Five banks offer nominal rates of 6 percent on deposits, but A pays interest annually, B pays semiannually, C quarterly, D monthly, and E daily.
   (1) What effective annual rate does each bank pay? If you deposited $5,000 in each bank today, how much would you have at the end of 1 year? 2 years?
   (2) If the banks were all insured by the government (the FDIC) and thus equally risky, would they be equally able to attract funds? If not, and the TVM were the only consideration, what *nominal rate* would cause all the banks to provide the same effective annual rate as Bank A?
   (3) Suppose you don't have the $5,000 but need it at the end of 1 year. You plan to make a series of deposits, annually for A, semiannually for B, quarterly for C, monthly for D, and daily for E, with payments beginning today. How large must the payments be to each bank?
   (4) Even if the 5 banks provided the same effective annual rate, would a rational investor be indifferent between the banks?
l. Suppose you borrowed $15,000. The loan's annual interest rate is 8 percent, and it requires 4 equal end-of-year payments. Set up an amortization schedule that shows the annual payments, interest payments, principal repayments, and beginning and ending loan balances.

Integrated Case

First National Bank

2-42 **Time value of money analysis** You have applied for a job with a local bank. As part of its evaluation process, you must take an examination on time value of money analysis covering the following questions.

a. Draw time lines for (1) a $100 lump sum cash flow at the end of Year 2, (2) an ordinary annuity of $100 per year for 3 years, and (3) an uneven cash flow stream of −$50, $100, $75, and $50 at the end of Years 0 through 3.

b. (1) What's the future value of $100 after 3 years if it earns 10 percent, annual compounding?
(2) What's the present value of $100 to be received in 3 years if the interest rate is 10 percent, annual compounding?

c. What annual interest rate would cause $100 to grow to $125.97 in 3 years?

d. If a company's sales are growing at a rate of 20 percent annually, how long will it take sales to double?

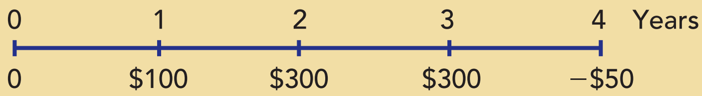
e. What's the difference between an ordinary annuity and an annuity due? What type of annuity is shown here? How would you change it to the other type of annuity?

f. (1) What is the future value of a 3-year, $100 ordinary annuity if the annual interest rate is 10 percent?
(2) What is its present value?
(3) What would the future and present values be if it were an annuity due?

g. A 5-year $100 ordinary annuity has an annual interest rate of 10 percent.
(1) What is its present value?
(2) What would the present value be if it was a 10-year annuity?
(3) What would the present value be if it was a 25-year annuity?
(4) What would the present value be if this was a perpetuity?

h. A 20-year-old student wants to save $3 a day for her retirement. Every day she places $3 in a drawer. At the end of each year, she invests the accumulated savings ($1,095) in a brokerage account with an expected annual return of 12 percent.
(1) If she keeps saving in this manner, how much will she have accumulated at age 65?
(2) If a 40-year-old investor began saving in this manner, how much would he have at age 65?
(3) How much would the 40-year-old investor have to save each year to accumulate the same amount at 65 as the 20-year-old investor?

i. What is the present value of the following uneven cash flow stream? The annual interest rate is 10 percent.



j. (1) Will the future value be larger or smaller if we compound an initial amount more often than annually, for example, *semiannually*, holding the stated (nominal) rate constant? Why?
(2) Define (a) the stated, or quoted, or nominal, rate, (b) the periodic rate, and (c) the effective annual rate (EAR or EFF%).
(3) What is the EAR corresponding to a nominal rate of 10 percent compounded semiannually? Compounded quarterly? Compounded daily?
(4) What is the future value of $100 after 3 years under 10 percent semiannual compounding? Quarterly compounding?

k. When will the EAR equal the nominal (quoted) rate?

l. (1) What is the value at the end of Year 3 of the following cash flow stream if interest is 10 percent, compounded semiannually? (Hint: You can use the EAR and treat the cash flows as an ordinary annuity or use the periodic rate and compound the cash flows individually.)

Periods	0	1	2	3	4	5	6
	0		$100		$100		$100

(2) What is the PV?

(3) What would be wrong with your answer to parts l(1) and l(2) if you used the nominal rate, 10 percent, rather than either the EAR or the periodic rate, $I_{NOM}/2 = 10\%/2 = 5\%$ to solve them?

m. (1) Construct an amortization schedule for a $1,000, 10 percent annual interest loan with 3 equal installments.

(2) What is the annual interest expense for the borrower, and the annual interest income for the lender, during Year 2?

cyberproblem

Please go to the ThomsonNOW Web site to access the Cyberproblems.